ELIU's Calculus Tutorial for AP Physics C

1. Review Precalculus H Ch 13. See notes on class website.
   - Limits
   - Continuity
   - Differentiable = Locally Linear
   - Average rate of change = slope of the secant line = \( \frac{f(b) - f(a)}{b - a} = \frac{\Delta f}{\Delta t} \)
   - Instantaneous rate of change = slope of tangent line = derivative = \( f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \)

Riemann Sum

2. The Shortcut Method. Memorize these formulas for derivatives:
   - \( \frac{d}{dx} C = 0 \) \( \frac{d}{dx} C = 0 \) (C = constant)
   - \( \frac{d}{dx} x^n = nx^{n-1} \)
   - \( \frac{d}{dx} \sin x = \cos x \)
   - \( \frac{d}{dx} \cos x = -\sin x \)
   - \( \frac{d}{dx} e^x = e^x \)
   - \( \frac{d}{dx} \ln x = \frac{1}{x} \)

First memorize the derivatives in #2. Then the antiderivatives are just going backward.

3. The antiderivative of function \( f(x) \) is another function \( F(x) = \int f(x) \, dx \)
   - whose derivative is \( f(x) \).
   - Example: An antiderivative of \( f(x) = x^3 \) is \( \int x^3 \, dx = \frac{x^4}{4} + C \).
   - Since \( \frac{d}{dx} \left( \frac{x^4}{4} + C \right) = \frac{4x^3}{3} + 0 = x^3 \), \( \frac{x^3}{3} + 1 \) is also an antiderivative of \( x^3 \).
   - The general antiderivative of \( f(x) = x^n \) is \( \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \) where \( C \) is any real number constant.
   - So going backward, memorize these antiderivatives:
     - \( \int dx = x + C \)
     - \( \int x \, dx = \frac{x^2}{2} + C \)
     - \( \int e^x \, dx = e^x + C \)
     - \( \int \sin x \, dx = -\cos x + C \)
     - \( \int \cos x \, dx = \sin x + C \)
     - \( \int \ln x \, dx = \frac{1}{x} + C \)
The definite integral of $f(x)$ on $[a, b]$ is the signed area between the curve and the x-axis on $[a, b]$. It's also found by the Riemann Sum, using approximations.

The notation for this definite integral comes from the definite integral of a function, $\int_a^b f(x) \, dx$. The definite integral is the limit of the Riemann Sum as the number of rectangles approaches infinity.

More on Derivatives

$\frac{d}{dx} \left( e^{x} + \sin x \right) = e^{x} x \cdot \cos x$
5. More on Derivatives (continued)

* The Chain Rule (like peeling a cabbage or onion) 

\[
\frac{d}{dx} f(g(x)) = f'(g(x)) g'(x)
\]

derviative of the outer function at the inner function times the derivative of the inner function.

Example:

\[
\frac{d}{dx} \sin(x^3) = \cos(x^3) \cdot (3x^2)
\]

\[
\frac{d}{dx} (\sin x)^3 = 3(\sin x)^2 (\cos x)
\]

\[
\frac{d}{dx} (\ln x) = \frac{1}{x} \cdot (\ln x)^{-2} = -x \ln^2 x
\]

\[
\frac{d}{dx} (\ln(x^2 \cos x)) = \frac{1}{x^2 \cos x} \cdot (2x \cos x + x^2 \sin x)
\]

\[
\frac{d}{dx} e^{x \ln x} = e^{x \ln x} \cdot (x \ln x)' = e^{x \ln x} (\ln x + \frac{x}{x}) = e^{x \ln x} (\ln x + 1)
\]

6. More on Integrals: u-substitution

Example:

\[
\int \sin^2 x \cos x \, dx.
\]

Whose derivative is \(\sin^2 x \cos x\)?

Let \(u = \sin x\) (Pick u wisely)

\[
du = \cos x \, dx
\]

This is the differential of \(u\). It is defined as \(du = \frac{du}{dx} \, dx\)

Plug in Substitute

\[
\int u^2 \, du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C
\]

Check:

\[
\frac{d}{dx} (\frac{\sin^3 x}{3} + C) = \frac{3\sin^2 x \cos x}{3} + 0 \equiv \text{same}
\]

Example:

\[
\int \cos(x^3) \cdot x^2 \, dx
\]

Let \(u = x^3\)

\[
du = 3x^2 \, dx
\]

Then

\[
\frac{1}{3} \int \cos u \, du = \frac{1}{3} \sin u + C = \frac{1}{3} \sin(x^3) + C
\]

Check:

\[
\frac{d}{dx} (\frac{\sin(x^3)}{3} + C) = \frac{3 \cos(x^3) \cdot 3x^2}{3} \equiv \text{same}
\]

So

\[
\int_{-2}^{5} \cos(x^3) \cdot 3x^2 \, dx = \left[ \frac{\sin(x^3)}{3} \right]_{-2}^{5}
\]

\[
= \frac{\sin(125)}{3} - \frac{\sin(-8)}{3} = 0.124
\]

More on Integrals: Linear too

\[
\int k f(x) \, dx = k \int f(x) \, dx
\]

Pull out constant

\[
\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx
\]

NO PRODUCT / QUOTIENT RULE for integrals!
More on Integrals.

When you are comfortable with u-substitution, you can skip explicitly writing \( u \, du \). Just group the \( du \) in parentheses, balance as needed.

Example:

\[
\int \left( x^4 + 3x \right) \left( 4x^3 + 3 \right) \, dx = \frac{\left( x^4 + 3x \right)^4}{4} + C
\]

That's \( u \), this must be \( du \)

\[
\int (\sin \theta - \cos \theta) \, d\theta = -\cos \theta - \sin \theta + C
\]

\[
\int \frac{3y}{\sqrt{2y^2 + 5}} \, dy = \frac{3}{2} \sqrt{2y^2 + 5} + C
\]

\[
\frac{3y}{\sqrt{2y^2 + 5}} \left( \frac{3}{2} \right) \int dy = \frac{3y}{2} \sqrt{2y^2 + 5} + C
\]

Balance

\[
= \frac{3y}{2} \sqrt{2y^2 + 5} + C
\]

Mean Value Theorem (MVT) for Derivatives

If \( f(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\), then \( \exists c \in (a, b) \) st.

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

Easy to understand with a picture.

\( f \) is a smooth (no sharp corners) string.
There must be at least one place \( c \) between \( a \) and \( b \) where the slope of the tangent at \( c \) is the same (parallel) as the slope of the secant from \( a \) to \( b \).

Some Important formulas in Physics.

Understand (as discussed in Physics B) and Memorize.

Position \( x(t) \)

Velocity \( v(t) = \frac{dx}{dt} \)

Acceleration \( a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} \)

\( F = ma \)

\( F = \int x(t) \, dx \) \( \Rightarrow \) \( \Delta W = \int_a^b F(x) \, dx \)

\( \frac{dW}{dt} = \frac{d}{dt} \int_a^b F(x) \, dx \)

Power \( P = \frac{dW}{dt} \) — how fast work is done

\( \Rightarrow \) since \( \frac{dW}{dt} = \int_a^b F(x) \, dx \)

Mean Value Theorem for Integrals

\( f \) is continuous on \([a, b]\)

\( \Rightarrow \exists c \in (a, b) \) st.

\[
\int_a^b f(x) \, dx = f(c)(b - a)
\]

Easy to understand with a picture:

There's a point \( c \) between \( a \) and \( b \) where the area of the rectangle at \( c \) equals the area under the curve.

Also defined:

\[
\bar{f} = \frac{1}{b - a} \int_a^b f(x) \, dx
\]

as the average of \( f \) on \([a, b]\).
9. About differentials \( dy \)
(used in physics for error analysis)

The slope of the tangent line at \( x_0 \) is
\[
f'(x_0) = \frac{df}{dx} = \frac{\text{rise}}{\text{run}}, \quad \text{Run} = \frac{dx = \Delta x}{x_0}
\]
So rise = slope \times \text{run}, for the independent variable
\[
\Delta f = f'(x_0) \Delta x \approx df
\]

The actual change in \( f \) as \( x \) goes from \( x_0 \) to \( x_0 + \Delta x \)
is \( \Delta f = f(x_0 + \Delta x) - f(x_0) \).
The change in the tangent line is \( df \), but \( df \approx \Delta f \) and are close in value when \( \Delta x \) is small.

Example: \( \frac{dy}{dx} = \frac{1}{x} \)  
Find the solution such that when \( x = 2, y = 2 \)

For very simple cases, take the antiderivative

\[
y(x) = \ln x + C \leftarrow \text{general solution}
\]

\[
y(e) = 2
\]

\[
2 = \ln e + C \Rightarrow C = 1
\]

\[
\therefore y(x) = \ln x + 1 \quad \text{is the particular solution}
\]

Another method to find \( y \) is to separate the variables:
\[
\frac{dy}{y} = \frac{1}{x} \, dx
\]
Then take the antiderivative on both sides.
\[
\int dy = \int \frac{1}{x} \, dx
\]
\[
yC_1 = \ln x + C_2
\]
\[
y = \ln x + (C_2 - C_1) \quad \text{Since the } C \text{'s are general constants, you can just put them together}
\]

10. Introduction to Differential Equations (DE)

\[
\frac{dy}{dx} = 2x
\]
A differential equation, the solution to this equation is not a number but rather a family of functions.
The solution is the set of all functions \( y(x) \) that make the equation true, so it's the functions whose derivative is \( 2x \).

\[
y(x) = x^2 + C
\]
The "general" solution, since the derivative is \( 2x \)

"General" means +C, \( C \) any real constant

\[
y(x) = x^2 + 2
\]
A particular solution.
Example: A particle's acceleration is given as \(a(t) = (2t+3)^{-3}\) m/s² (motion along a line). The initial velocity is \(v(0) = 4\) m/s. Find the velocity as a function of time.

* In physics C, you will see many differential equations. You generally will need to separate the variables. The equations may be solved by the "+ C" indefinite integral method, but it's generally faster to take definite integrals with matching \(0^\text{th}\) values on both sides.

**Indefinite Integral Method**

\[ \frac{dv}{dt} = (2t+3)^{-3} \]

\[ dv = (2t+3)^{-3} \; dt \]

\[ \int dv = \int (2t+3)^{-3} \; dt \]

\[ \int dv = \frac{1}{2} (2t+3)^{-2} + C \]

\[ v(t) = \frac{1}{2} (2t+3)^{-2} + C \]

**Definite integral method**

\[ \int_0^t dv = \frac{1}{2} \left[ (2t+3)^{-2} \right]_0^t \]

\[ v(t) = \frac{1}{2} \left[ (2t+3)^{-2} - 1 \right] \]

\[ v(t) = \frac{1}{2} \left[ (2t+3)^{-2} - 1 \right] \]

\[ \frac{dv}{dt} = \frac{1}{3} (3x+9x^2 + 0) \]

\[ \frac{dy}{dx} = \frac{x+3x^2}{y^2} \quad y = 6 \quad \text{when} \quad x = 0. \]

**Method 1:** Antiderivative? Not that simple.

**Method 2:** Separate the variables.

\[ y^2 \frac{dy}{dx} = (x+3x^2) \; dx \]

\[ \int y^2 \frac{dy}{dx} = \int (x+3x^2) \; dx \]

\[ \left[ \frac{y^3}{3} + C_1 \right] = \frac{x^2}{2} + x^3 + C_2 \]

\[ \frac{y^3}{3} = \frac{x^2}{2} + x^3 + (C_2 - C_1) \]

\[ \frac{y^3}{3} = \frac{3}{2} x^2 + 3x^3 + \frac{3(C_2 - C_1)}{C} \]

\[ y = \sqrt[3]{ \frac{3}{2} x^2 + 3x^3 + C } \quad \text{General solution} \]

Plug initial condition, \(x = 0, y = 6\)

\[ 6 = \sqrt[3]{ \frac{3}{2} (0)^2 + 3(0)^3 + C } \Rightarrow C = 6^3 = 216 \]

\[ y = \sqrt[3]{ \frac{3}{2} x^2 + 3x^3 + 216 } \quad \text{Particular solution} \]

Check:

\[ \frac{dy}{dx} = \frac{1}{3} (3x+9x^2 + 216)^{-2/3} (3x+9x^2 + 0) \]

\[ = \frac{x+3x^2}{(x+3x^2)^2} \quad \text{same} \]
Example: RC Circuit
A resistor, capacitor, battery, and switch are connected in series. At time \( t=0 \), the switch is closed.
Find, \( q(t) \), charge on the capacitor as a function of time.
As discussed in AP Physics B, the graph is

\[
q(t) = \begin{cases} \text{Constant} & \text{for } t < 0 \\ \frac{1}{RC} \int_0^t (-\frac{1}{C} \frac{dq}{dt}) dt & \text{for } t \geq 0 \end{cases}
\]

\[-\int_0^t \frac{1}{CE-q} (-\frac{dq}{dt}) dt = \int_0^t \frac{1}{RC} dt\]
\[-\ln(CE-q) \Big|_0^t = \frac{1}{RC} \int_0^t dt\]
\[-(\ln(CE-q) - \ln(CE)) = \frac{t}{RC}\]
\[\ln \left( \frac{CE-q}{CE} \right) = -\frac{t}{RC}\]
\[\frac{CE-q}{CE} = e^{-\frac{t}{RC}}\]
\[CE-q = CE - \frac{CE}{e^{\frac{t}{RC}}}\]

\[q(t) = CE \left( 1 - e^{-\frac{t}{RC}} \right)\]

Mathematical derivation is below:
Kirchhoff's Loop Rule
\[E - iR - \frac{q}{C} = 0\]
\[E - \frac{dq}{dt} R - \frac{q}{C} = 0\]
\[E - \frac{dq}{dt} R - \frac{q}{C} = 0\] is a differential equation with initial condition, \( q = 0 \) at \( t = 0 \)

Separate the variables
\[R \frac{dq}{dt} = E - \frac{q}{C}\]
\[\frac{dq}{dt} = \frac{1}{RC} (CE-q)\]
\[\frac{1}{CE-q} dq = \frac{1}{RC} dt\]
Use the definite integral method
\[\int_0^t \frac{1}{CE-q} dq = \int_0^t \frac{1}{RC} dt\]

II We will start AP Physics C Electromagnetism in December. The calculus needed will require mastery of volume integrals, which we will combine with use of vectors. In AP Calculus AB, be extra studious when learning about Riemann Sums and Ch. 6 Volume Integrals. In particular, become comfortable with the idea of slice, approximate, integrate.
A amount of charge is uniformly distributed around a conducting ring of radius \( R \).

Find the electric field \( E(x) \) at a distance \( x \) from the center of the ring.

**Approximate** the electric field due to a slice (shaded piece).

\[
\Delta E = k \frac{4\pi \sigma}{r^2} \cos \Theta
\]

The \( x \) components from the two opposite shaded regions add up while the \( \sin \Theta \) components cancel out.

\[
= k \frac{\sigma}{2\pi R} \frac{dx}{(\sqrt{x^2+R^2})^2} \cdot \frac{x}{\sqrt{x^2+R^2}} \cdot \cos \Theta
\]

\[
dE = \frac{4\pi \sigma}{2\pi R} \frac{x}{(x^2+R^2)^{3/2}} \, ds
\]

Integrate over all slices,

\[
E = \frac{1}{4\pi \varepsilon_0} \frac{\sigma}{2\pi R} \frac{x}{(x^2+R^2)^{3/2}} \int ds
\]

\[
= \frac{Qx}{4\pi \varepsilon_0 (x^2+R^2)^{3/2}}
\]