Lesson #8 Handout (8/31/2009)

I. Sequences and Series

II. Counting

III. Inequalities and Equalities

**Sequence**: A function with domain consisting of the natural numbers (or integers)

\[ \{ \chi_n \} : n \in \mathbb{N} \]

\[ f(n) = \chi_n, \quad n = 0, 1, 2, 3, \ldots \]

**Convergence**: The sequence \( \{ \chi_n \} \) in the real number system is said to converge if there exists a number \( p \in \mathbb{R} \) such that:

- \( \forall \varepsilon > 0, \exists N \) integer such that, \( |\chi_n - p| < \varepsilon \) whenever \( n \geq N \)

- We say \( \{ \chi_n \} \) converges to \( p \)

\[ \lim_{n \to \infty} \chi_n = p \]

It is not enough to say that "as \( n \) gets bigger, \( \chi_n \) gets closer to \( p \)."

In the example of the picture, \( \chi_{n+1} \) is not necessarily nearer \( p \) than \( \chi_n \), so "\( n \) is bigger" doesn't mean "\( \chi_n \) is closer to \( p \)." But \( \{ \chi_n \} \) does converge to \( p \).

I have any requirement of "nearness." No matter how "near" the sequence should be near \( p \) (choose \( \varepsilon > 0 \)), \( \chi_n \) will be within \( \varepsilon \) of \( p \) as long as \( n \geq N \) (\( n \) is big enough).

**Series**: A sequence of partial sums (Fourier Series, Taylor Series)

\[ S_n = \sum_{i=1}^{n} \chi_i \]

the \( n \)th partial sum

"A series converges" means

\[ \lim_{n \to \infty} S_n = S \]

Useful Series:

\[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]

\[ \frac{n(n+1)(2n+1)}{6} \quad \text{is proof by collapsing sum} \]

\[ \frac{n(n+1)(2n+1)}{6} = \sum_{i=1}^{n} i^2 = \frac{1}{2} n^3 + \frac{3}{2} n^2 + \frac{1}{2} n \]

**Geometric Series**

\[ \sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r} \]

always ok \( 1 - r \neq 0 \)

\[ \sum_{i=0}^{n} r^i = \frac{1}{1-r} \]

converges if \( |r| < 1 \)

\[ r^0 + r^1 + r^2 + \ldots + r^n = S_n \]

\[ r^0 + r^1 + r^2 + \ldots = S \]

Change of index: Let \( j = i \)

\[ \sum_{i=0}^{n} \chi_i \]

\[ \sum_{j=1}^{n+1} \chi_j \]
Arithmetic Sequence: $a_n - a_{n-1} = d$ constant difference

$a_1, a_1 + d, a_1 + 2d, \ldots$

$S_n = \sum_{n=1}^{N} a_n = \sum_{n=1}^{N} [a_1 + (n-1)d]$

$= a_1N + \frac{N(N+1)d}{2}$

**Nth partial sum**

- The number between two terms of an arithmetic sequence is the arithmetic mean

$$\frac{x}{2} = \frac{y}{2}$$

$$a_1, a_1 + d, a_1 + 2d$$

$$? = a_1 + d = \frac{a_1 + (a_1 + 2d)}{2}$$

**Geometric Sequence: $a_n = a_1r^{n-1}$ constant ratio**

$a_1, ra_1, r^2a_1, \ldots$

- The Nth partial sum is that of a geometric series

$$S_n = a_1(1 + r + r^2 + \ldots + r^n) = \frac{a_1(1 - r^{n+1})}{1 - r}$$

- The sum of the infinite series is

$$a_1(1 + r + r^2 + \ldots) = \frac{a_1}{1 - r}$$ Only converges for $|r| < 1$

- The number between two terms of a geometric sequence is the geometric mean $\sqrt{xy}$

$$x, \frac{?}{2}, y$$

$$a_1, ra_1, r^2a_1$$

$$? = ra_1 = \sqrt[3]{a_1r^2a_1}$$

**Recursive Sequence**

(a) $a_1 = 3$

$$a_n = 2a_{n-1} + 5$$

Find $a_4$

$$\frac{a_1}{3} = 2 \frac{a_2}{3} = 5$$

- (b) $a_1 = 1, a_3 = 1$

$$a_n = a_{n-1} + a_{n-2}, n \geq 3$$

Find $1st 7$ terms

$\text{Soln: } a_1 = 1, 2, 3, 5, 8, 13$

**First term of geometric series is 64**, common ratio $\frac{1}{4}$. For what $n$ is $t_n = \frac{1}{4}$?

- Soln:

$$t_1 = 64$$

$$t_n = 64 \left(\frac{1}{4}\right)^{n-1} \Rightarrow \frac{1}{4}$$

$$t_n = 4^n \Rightarrow \frac{4^n}{4}$$

$$n = 5$$

**Sequence**

1st 3 terms: arithmetic

2, x, y, 9

Last 3 terms: geometric

$x, y, 9$

**Method 2: (longer way)**

- $x + y = 2 + d$

- $y = 2 + 2(x - 2)$

- $9 + \left(\frac{4}{x}\right)^2$
1. In an arithmetic series, $S_n = 3n^2 + 2n$

Find the first three terms.

Solution: arithmetic sequence $a_1, a_2, a_3, \ldots, a_n = a_1 + (n-1)d$

arithmetic series $a_1 + a_2 + a_3 + \ldots$

$a_1 = S_1 = 3(1)^2 + 2(1) = 5$

$S_2 = a_2 = 3(2)^2 + 2(2) = 16 \Rightarrow a_2 = 11 \Rightarrow d = 6$

$a_3 = a_2 + d = 11 + 6 = 17$

2. $1, 2, 4, \ldots$ Geometric sequence

Sum of first 7 terms?

Solution: $r = \frac{4}{2} = 2$,

$a_1 + ra_1 + r^2a_1 + \ldots, + r^6a_1 = a_1 \frac{1-r^7}{1-r} = 1 \frac{1-2^7}{1-2} = 127$

3. Repeating decimal

$0.4545\ldots$ as fraction?

Solution: $\frac{45}{100} + \frac{45}{100^2} + \ldots = 45 \left[ \frac{1}{100} \right] = 45 \left[ \frac{1}{1-\frac{1}{100}} \right]$

Or, $r = \frac{1}{100}, a_1 = \frac{45}{100}$

$a_1 \frac{1-r^2}{1-r} = 45 \left[ \frac{1}{1-\frac{1}{100}} \right] = \frac{45}{99} = \frac{5}{11}$

Check with a calculator

$0.4545454545 \rightarrow \frac{5}{11}$

$\frac{7177}{10000} + \frac{7177}{1000^2} + \ldots = \frac{7177/1000}{1 - \frac{1}{1000}} = \frac{7177}{9999}$

4. Counting

- Factorials: $n!$ "n prime", "n factorial"

$n! = n(n-1)(n-2)\ldots2\cdot1$ (n an integer)

5. $(n+2)-(n+1)!$

Simplify $n!$

Solution: $(n+2)(n+1) - (n+1) = (n+1)(n+2-1) = (n+1)^2$

6. $2 + (-\frac{1}{2}) + \frac{1}{8} + (-\frac{1}{32}) + \ldots = ?$

Solution: $-\frac{1}{2} = -\frac{1}{4}, \frac{1}{8} = -\frac{1}{4}$

Geometric series

$a_1 = 2, r = -\frac{1}{4}, a_{\infty} = a_1 \frac{1}{1-r} = \frac{2}{1-\frac{1}{4}} = \frac{8}{3}$

7. Venn Diagrams

$\left( E \cap F \cap G \right) \cup \text{UF}$

8. Among seniors, 80 take math, 11 take Spanish, 54 take physics. 10 take math and Spanish; 19 take math & physics & Spanish. Seven take all three. How many seniors take math but not Spanish or physics?

Solution: $M \cap S = 80 - (3+7+12) = 58$

9. total $80$, $80 - (3+7+12) = 58$
There are 50 pets. 28 are birds, and 32 are under the age of 1 year. 12 are birds under the age of 1 year. How many lizards over the age of 1 year are in the room?

Solution 1: Total 50 pets

Under

<table>
<thead>
<tr>
<th>12</th>
<th>B</th>
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Over

<table>
<thead>
<tr>
<th>28</th>
</tr>
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<tbody>
<tr>
<td>32</td>
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16 + 2 = 18

Solution 2:

\[
\begin{align*}
\text{B} & \quad \text{Lizards over 1 year old} \\
\text{B} & \quad 12 \\
\text{L} & \quad 28 \\
\end{align*}
\]

\[
50 - (28 + 32 - 12) = 50 - 48 = 2
\]

**Generalized Counting Principle**

\[E_1, E_2, \ldots, E_k\] are sets with \[n_1, n_2, \ldots, n_k\] elements respectively.

\[\Rightarrow n_1 \times n_2 \times \ldots \times n_k\] ways to choose one element from \(E_1\), then one element from \(E_2,\ldots\)

There are \(n\) routes from Town A to B and \(m\) routes from Town B to C. For each way you get from A to B, there are \(n\) ways to get from B to C, so there are \(n \times m\) ways to go from A to C via Town B.

**Permutation:** \(nPr\). There are \(n\) different objects. Arrange \(r\) of them, where order matters (on a bookshelf).

\[
\text{Arrange } r \text{ of them, where order matters (on a bookshelf)}
\]

\[
\frac{n!}{(n-r)!} \cdot \frac{n-r}{r}
\]

20 members in team. Choose one president, one vice president, one treasurer. How many ways?

Solution: 20 \(\times\) 19 \(\times\) 18

For each way you choose a president, there are 19 ways you can choose a VP. And for each prez and VP, there are 18 ways to choose a treasurer.
Power Set of A is the set of all subsets of A. If A has \( n \) elements, its power set has \( 2^n \) elements.

Each subset of A either has or does not have an element in A. If yes or no each subset has \( 2^n \) elements.

There are \( n \) different objects. Take \( r \) of them, where order does not matter (without replacement). There are \( nC_r \) ways to do this.

\[
 nCr = \binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n!}{n-r} r!
\]

\[
 \binom{n}{2} = \binom{n}{n-1} = 1
\]

Combinations

Repetitions

\( n = n_1 + n_2 + \ldots + n_k \). There are \( n \) objects of \( k \) different types, where \( n_1 \) objects are type 1, \( n_2 \) are type 2, \ldots, \( n_k \) are type \( k \).

The number of distinguishable permutations is

\[
\frac{n!}{n_1!n_2!\ldots n_k!}
\]

**Ex.** 20 members in team. Make a 3-person committee. How many possible committees?

**Soln:** Unlike (ex.12), order doesn't matter.

If Mary, Alice, and Bill are on the committee, it doesn't matter if it's MBA, MBA, AMB, etc.

\[
\binom{20}{3} = \frac{20!}{17!3!} = \frac{20 \times 19 \times 18}{3 \times 2} = 1140
\]

**Ex.** (A) How many permutations of the letters in STANFORD?

**Soln:** 8 different letters, \( 8! \) distinguishable perms.

(B) How many permutations of BERKELEY, call it P?

**Soln:** The 3 E's are indistinguishable.

\[
8! \text{ counts } \{ \text{BYE, RE}_2 \text{ LE}_2 \text{ K} \} \text{ BYE}_2 \text{ RE}_3 \text{ LE}_3 \text{ K}
\]

These as \( \{ \text{BYE}, \text{RE}_3 \text{ LE}_3 \text{ K} \} \text{ BYE}_3 \text{ RE}_2 \text{ LE}_2 \text{ K} \) different.

Wherever the 3 E’s are, there are 3! indistinguishable permutations.

So \( 8! = P \cdot 3! \Rightarrow P = \frac{8!}{3!} \)

**Ex.** 15 How many distinguishable permutations of letters in BOOKSHOPS? 9 letters, 3 O's, 2 S's

\[
\frac{9!}{3!2!} = 30,240
\]
There is an unlimited supply of 6 colors of candy. Choose 3 candies and put them in a bag. How many 3-color combinations could be in the bag?

Solution (Soln): Imagine 5 circles and \( n - 1 \) sticks:

- 0001 means the bag has 3 candies of color 1 and no candy of any other color.
- 00110 means the bag has 1 color-1 candy, 1 color-2 candy, and 1 color-5 candy.

The 5 sticks separate the 3 circles into 6 compartments. If a ball is in compartment 1, it means a candy of color 1 was chosen. There are how many permutations of the 3 circles and 5 sticks?

\[
\text{Permutations} = \frac{8!}{(3!)(5!)} = \frac{8!}{(3!)(5!)} = 56
\]

There are 2 red chairs and 4 blue chairs. How many ways can you arrange them in a row?

Solution (Soln):

- \( \frac{6!}{(4!)(2!)} \) ways to arrange the chairs, (i.e., arrange the blue chairs)

A bit consists of a 0 or 1. How many \( n \)-bit codewords can you make?

Solution (Soln):

\[
\text{Codewords} = \frac{2^n}{n}
\]