Theorem 8.1:

Necessity: If $\underline{x}_{LS}$ is optimal, then

$$\|\underline{A}\underline{x}_{LS} - \underline{b}\|_2^2 \leq \|\underline{A}\underline{x} - \underline{b}\|_2^2$$

for any $\underline{x} \in \mathbb{C}^n$. Suppose

$$\underline{x} = \underline{x}_{LS} + \alpha \underline{z}$$

where $\alpha > 0$, and

$$\underline{z} = -\underline{A}^H(\underline{A}\underline{x}_{LS} - \underline{b}) \neq \underline{0}.$$

Then,

$$\begin{aligned}
\|\underline{A}\underline{x} - \underline{b}\|_2^2 &= \|\underline{A}\underline{x}_{LS} - \underline{b} + \alpha \underline{A}\underline{z}\|_2^2 \\
&= \|\underline{A}\underline{x}_{LS} - \underline{b}\|_2^2 + 2\alpha \,\mathrm{Re}\{\underline{z}^H \underline{A}^H(\underline{A}\underline{x}_{LS} - \underline{b})\} + \alpha^2 \|\underline{A}\underline{z}\|_2^2 \\
&= \|\underline{A}\underline{x}_{LS} - \underline{b}\|_2^2 - 2\alpha \|\underline{z}\|_2^2 + \alpha^2 \|\underline{A}\underline{z}\|_2^2
\end{aligned}$$

Now, choose

$$\alpha < 2\|\underline{z}\|_2^2 / \|\underline{A}\underline{z}\|_2^2 .$$

Then,

$$\|\underline{A}\underline{x} - \underline{b}\|_2^2 < \|\underline{A}\underline{x}_{LS} - \underline{b}\|_2^2$$

which is a contradictory statement unless $\underline{z} = \underline{0}$.

Sufficiency:

We can express any $\underline{x} \in \mathbb{C}^n$ as

$$\underline{x} = \underline{x}_{LS} + \underline{z}$$

where $\underline{x}_{LS}$ satisfies $\underline{A}^H(\underline{A}\underline{x}_{LS} - \underline{b}) = \underline{0}$. Then,

$$\begin{aligned}
\|\underline{A}\underline{x} - \underline{b}\|_2^2 &= \|\underline{A}\underline{x}_{LS} - \underline{b} + \underline{A}\underline{z}\|_2^2 \\
&= \|\underline{A}\underline{x}_{LS} - \underline{b}\|_2^2 + 2\,\mathrm{Re}\{\underline{z}^H\underline{A}^H(\underline{A}\underline{x}_{LS} - \underline{b})\} + \|\underline{A}\underline{z}\|_2^2 \\
&= \|\underline{A}\underline{x}_{LS} - \underline{b}\|_2^2 + \|\underline{A}\underline{z}\|_2^2 \\
&\geq \|\underline{A}\underline{x}_{LS} - \underline{b}\|_2^2
\end{aligned}$$

which means that $\underline{x}_{LS}$ is a solution.

Theorem 8.2:

For any $\underline{x} \in \mathbb{C}^n$,

$$\|A x - b\|_2^2 = \cancel{\| U^H(A x V V^H b - b)\|}$$
$$= \| \underline{U}^H(\underline{A}\underline{V}\underline{V}^H\underline{x} - \underline{b})\|_2^2$$
$$= \left\| \begin{bmatrix} \underline{\Sigma} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{d}_1 \\ \underline{d}_2 \end{bmatrix} - \begin{bmatrix} \underline{c}_1 \\ \underline{c}_2 \end{bmatrix} \right\|_2^2 \qquad (1)$$

where

$$\underline{d} = \underline{V}^H\underline{x} = \begin{bmatrix} \underline{V}_1^H\underline{x} \\ \underline{V}_2^H\underline{x} \end{bmatrix} = \begin{bmatrix} \underline{d}_1 \\ \underline{d}_2 \end{bmatrix}$$

$$\underline{c} = \underline{U}^H\underline{b} = \begin{bmatrix} \underline{U}_1^H\underline{b} \\ \underline{U}_2^H\underline{b} \end{bmatrix} = \begin{bmatrix} \underline{c}_1 \\ \underline{c}_2 \end{bmatrix}$$

We further express (1) as:

$$\|\underline{A}\underline{x} - \underline{b}\|_2^2 = \left\| \begin{bmatrix} \underline{\Sigma}\underline{d}_1 - \underline{c}_1 \\ \underline{c}_2 \end{bmatrix} \right\|_2^2$$
$$= \|\underline{\Sigma}\underline{d}_1 - \underline{c}_1\|_2^2 + \|\underline{c}_2\|_2^2$$
$$\geq \|\underline{c}_2\|_2^2$$

Equality in the above equation is achieved when $\underline{\Sigma}\underline{d}_1 - \underline{c}_1 = \underline{0}$,

or

$$\underline{V}_1^H\underline{x} = \underline{\Sigma}^{-1}\underline{U}_1^H\underline{b} \qquad (2)$$

Consider the 2-norm of $\underline{x}$ that satisfies (2).

$$\|\underline{x}\|_2^2 = \|\underline{V}^H\underline{x}\|_2^2$$

$$= \|\underline{V}_1^H\underline{x}\|_2^2 + \|\underline{V}_2^H\underline{x}\|_2^2 \tag{3}$$

Under the condition in (2), the first term in (3) is fixed. Hence, to minimize $\|\underline{x}\|_2^2$ subject to (2), we can have

$$\underline{V}_2^H\underline{x} = \underline{0}.$$

Subsequently,

$$\underline{V}^H\underline{x} = \begin{bmatrix} \underline{V}_1^H\underline{x} \\ \underline{V}_2^H\underline{x} \end{bmatrix} = \begin{bmatrix} \underline{\Sigma}^{-1}\underline{U}_1^H\underline{b} \\ \underline{0} \end{bmatrix}$$

$$\Longleftrightarrow \quad \underline{x} = \underline{V} \begin{bmatrix} \tilde{\underline{\Sigma}}^{-1}\underline{U}_1^H\underline{b} \\ \underline{0} \end{bmatrix}$$

$$= \underline{V}_1 \tilde{\underline{\Sigma}}^{-1}\underline{U}_1^H\underline{b}.$$