LECTURE 7:

Property 7.1: left to you as an ~~exercise~~ exercise.

Property 7.2: Let $\underline{C} = [\underline{c}_1, \cdots, \underline{c}_n]$ be the inverse of $\underline{A}$.
The condition

$$\underline{A}\,\underline{C} = \underline{I}$$

is equivalent to

$$\underline{A}\,\underline{c}_k = \underline{e}_k \quad , \quad k = 1, \cdots, n \qquad (*)$$

Since $\underline{A}$ is lower Δ, $(*)$ can be solved using forward substitution. By forward substitution, one can show that

$$c_{ik} = 0 , \qquad i = 1, 2, \cdots, k-1.$$

thereby proving that $\underline{C}$ is lower Δ.
In addition, if $\underline{A}$ is lower Δ with diagonal unit entries, then $c_{kk} = 1$ meaning that $\underline{C}$ is lower Δ with unit diagonal entries.

Regarding the inverse result on p.1-6:

Consider $\underline{M}_1^{-1}\underline{M}_2^{-1}$:

$$\underline{M}_1^{-1}\underline{M}_2^{-1} = (\underline{I} + \underline{\tau}^{(1)}\underline{e}_1^T)(\underline{I} + \underline{\tau}^{(2)}\underline{e}_2^T)$$
$$= \underline{I} + \underline{\tau}^{(1)}\underline{e}_1^T + \underline{\tau}^{(2)}\underline{e}_2^T + \underline{\tau}^{(1)}\underbrace{\underline{e}_1^T\underline{\tau}^{(2)}}_{=0}\underline{e}_2^T$$

In a similar way, one can verify that

$$\prod_{i=1}^{n-1} M_i^{-1} = \underline{I} + \sum_{k=1}^{n-1} \underline{\tau}^{(k)}\underline{e}_k^T .$$

Theorem 7.1:

To show the existence of LU decomposition, ~~we look at~~ it suffices to find ~~situations~~ conditions where nonzero pivot elements $a_{kk}^{(k)}$ are guaranteed.
Consider the kth Gauss elimination

$$\underline{M}_k \underline{M}_{k-1} \cdots \underline{M}_1 \underline{A} = \underline{A}^{(k)} \qquad (*)$$

Let $\underline{L}^{(k)} = \underline{M}_k \underline{M}_{k-1} \cdots \underline{M}_1$. It is easily shown that $\underline{L}^{(k)}$ is lower triangular with unit diagonal entries.
Eq. $(*)$ takes on the structure

$$\begin{bmatrix} \times & \underline{0} \\ \times & \times \end{bmatrix} \begin{bmatrix} \underline{A}(\{1,\cdots,k\}) & \times \\ \times & \times \end{bmatrix} = \begin{bmatrix} \underline{A}_{11}^{(k)} & \times \\ \underline{0} & \times \end{bmatrix}$$

and thus

$$\underline{L}^{(k)}(\{1,\cdots,k\})\, \underline{A}(\{1,\cdots,k\}) = \underline{A}_{11}^{(k)}. \qquad (**)$$

The determinant of $(**)$ is

$$\begin{aligned} \det(\underline{A}_{11}^{(k)}) &= \det(\underline{L}^{(k)}(\{1,\cdots,k\}))\, \det \underline{A}(\{1,\cdots,k\}) \\ &= \det(\underline{A}(\{1,\cdots,k\})). \end{aligned} \qquad (***)$$

Since $\underline{A}_{11}^{(k)}$ is upper Δ, $\det(\underline{A}_{11}^{(k)}) = \prod_{i=1}^{k} a_{ii}^{(k)}$. Now, if $\det(\underline{A}(\{1,\cdots,k\})) \neq 0$, then $a_{kk}^{(k)} \neq 0$ must hold.

To prove uniqueness, suppose that $\underline{A}$ is nonsingular and has two LU decompositions $\underline{A} = \underline{L}_1 \underline{U}_1$ and $\underline{A} = \underline{L}_2 \underline{U}_2$. Then, it must be true that

$$\underline{L}_1^{-1} \underline{L}_2 = \underline{U}_1 \underline{U}_2^{-1} \qquad (*)$$

From Properties 7.1 and 7.2, $\underline{L}_1^{-1} \underline{L}_2$ is lower Δ with unit diagonal entries. As for $\underline{U}_1 \underline{U}_2^{-1}$, it can be shown (in a way similar to Properties 7.1 and 7.2) that $\underline{U}_1 \underline{U}_2^{-1}$ is upper Δ. This implies that $\underline{L}_1^{-1} \underline{L}_2 = \underline{I}$ and $\underline{U}_1 \underline{U}_2^{-1} = \underline{I}$, a contradiction to non-unique LU factors.

Property 7.3:

$$\begin{aligned} \det(\underline{A}) &= \det(\underline{L}\,\underline{U}) \\ &= \det(\underline{L}) \det(\underline{U}) \\ &= \det(\underline{U}) \\ &= \prod_{i=1}^{n} u_{ii} . \end{aligned}$$

Theorem 7.2

The matrix $\underline{M}^{-1}\underline{A}\,\underline{M}^{-H}$ is Hermitian. This implies

$$\underline{M}^{-1}\underline{A}\,\underline{M}^{-H} = \underline{M}^{-1}\underline{L}\,\underline{D}$$

is Hermitian. But $\underline{M}^{-1}\underline{L}$ is lower triangular with unit diagonal entries (Properties 7.1 and 7.2). Hence, we must have

$$\underline{M}^{-1}\underline{L} = \underline{I}.$$

Theorem 7.3

If $\underline{A}$ is PD, then it is nonsingular and has

$$\det(\underline{A}(\{1,\dots,k\})) > 0, \quad \forall\ k = 1,\dots,n$$

(cf. Property 5.1). Hence, from Theorem 7.1 there exists a unique LU decomposition or LDM decomposition for $\underline{A}$. Since $\underline{A}$ is Hermitian (by definition), we have from Theorem 7.2 that $\underline{L} = \underline{M}$. Furthermore, $\underline{D} = \operatorname{diag}(d_1,\dots,d_n)$ must satisfy $d_i > 0$ $\forall\ i$, since for any $\underline{x} \neq \underline{0}$,

$$\begin{aligned} 0 &< \underline{x}^H\underline{A}\,\underline{x} \\ &= \underline{x}^H\underline{L}\,\underline{D}\,\underline{L}^H\underline{x} \\ &= \sum_{i=1}^{n} d_i|z_i|^2, \qquad \underline{z} = \underline{L}^H\underline{x}. \end{aligned}$$

Let $\underline{D}^{1/2} = \mathrm{Diag}(\sqrt{d_1}, \cdots, \sqrt{d_n})$. The Cholesky factor is given by

$$\underline{G} = \underline{L}\,\underline{D}^{1/2} \quad .$$