

# LECTURE 7:

9.1

Property 7.1: left to you as an ~~exercise~~ exercise.

Property 7.2: Let  $\underline{C} = [\underline{c}_1, \dots, \underline{c}_n]$  be the inverse of  $\underline{A}$ .  
The condition

$$\underline{A}\underline{C} = \underline{I}$$

is equivalent to

$$\underline{A}\underline{c}_k = \underline{e}_k, \quad k=1, \dots, n \quad (*)$$

Since  $\underline{A}$  is lower  $\Delta$ ,  $(*)$  can be solved using forward substitution. By forward substitution, one can show that

$$c_{ik} = 0, \quad i=1, 2, \dots, k-1.$$

thereby proving that  $\underline{C}$  is lower  $\Delta$ .

In addition, if  $\underline{A}$  is lower  $\Delta$  with diagonal unit entries, then  $c_{kk}=1$  meaning that  $\underline{C}$  is lower  $\Delta$  with unit diagonal entries.

Regarding the inverse result on p. 16:

Consider  $\underline{M}_1^{-1}\underline{M}_2^{-1}$ :

$$\begin{aligned} \underline{M}_1^{-1}\underline{M}_2^{-1} &= (\underline{I} + \underline{r}^{(1)}\underline{e}_1^T)(\underline{I} + \underline{r}^{(2)}\underline{e}_2^T) \\ &= \underline{I} + \underline{r}^{(1)}\underline{e}_1^T + \underline{r}^{(2)}\underline{e}_2^T + \underbrace{\underline{r}^{(1)}\underline{e}_1^T \underline{r}^{(2)}\underline{e}_2^T}_{=0} \end{aligned}$$

In a similar way, one can verify that

$$\prod_{i=1}^{n-1} \underline{M}_i^{-1} = \underline{I} + \sum_{k=1}^{n-1} \underline{r}^{(k)}\underline{e}_k^T.$$

# Theorem 7.1:

To show the existence of LU decomposition, it suffices to find ~~we look at~~ conditions where nonzero pivot elements  $a_{kk}^{(k)}$  are guaranteed.

Consider the  $k$ -th Gauss elimination

$$\underline{M}_k \underline{M}_{k-1} \dots \underline{M}_1 \underline{A} = \underline{A}^{(k)} \quad (*)$$

Let  $\underline{L}^{(k)} = \underline{M}_k \underline{M}_{k-1} \dots \underline{M}_1$ . It is easily shown that  $\underline{L}^{(k)}$  is lower triangular with unit diagonal entries.

Eq. (\*) takes on the structure

$$\begin{bmatrix} \begin{array}{c|c} \triangle & 0 \\ \hline \square & \triangle \end{array} & \\ \hline & \begin{array}{c} \square \\ \triangle \end{array} \end{bmatrix} \begin{bmatrix} \underline{A}(\{1, \dots, k\}) & \times \\ \times & \times \end{bmatrix} = \begin{bmatrix} \underline{A}^{(k)} & \times \\ \underline{0} & \times \end{bmatrix}$$

and thus

$$\underline{L}^{(k)}(\{1, \dots, k\}) \underline{A}(\{1, \dots, k\}) = \underline{A}^{(k)} \quad (**)$$

The determinant of (\*\*) is

$$\begin{aligned} \det(\underline{A}^{(k)}) &= \det(\underline{L}^{(k)}(\{1, \dots, k\})) \det \underline{A}(\{1, \dots, k\}) \\ &= \det(\underline{A}(\{1, \dots, k\})) \end{aligned} \quad (***)$$

Since  $\underline{A}^{(k)}$  is upper  $\Delta$ ,  $\det(\underline{A}^{(k)}) = \prod_{i=1}^k a_{ii}^{(k)}$ . Now,

if  $\det(\underline{A}(\{1, \dots, k\})) \neq 0$ , then  $a_{kk}^{(k)} \neq 0$  must hold.

To prove uniqueness, suppose that  $A$  is nonsingular and has two LU decompositions  $A = L_1 U_1$  and  $A = L_2 U_2$ . Then, it must be true that

$$L_1^{-1} L_2 = U_1 U_2^{-1}$$

(\*)

From Properties 7.1 and 7.2,  $L_1^{-1} L_2$  is lower  $\Delta$  with unit diagonal entries. As for  $U_1 U_2^{-1}$ , it can be shown (in a way similar to Properties 7.1 and 7.2) that  $U_1 U_2^{-1}$  is upper  $\Delta$ . This implies that  $L_1^{-1} L_2 = I$  and  $U_1 U_2^{-1} = I$ , a contradiction to non-unique LU factors.

Property 7.3:

$$\begin{aligned} \det(A) &= \det(LU) \\ &= \det(L) \det(U) \\ &= \det(U) \\ &= \prod_{i=1}^n u_{ii} \end{aligned}$$

### Theorem 7.2

p. dx

The matrix  $\underline{M}^{-1} \underline{A} \underline{M}^{-H}$  is Hermitian. This implies

$$\underline{M}^{-1} \underline{A} \underline{M}^{-H} = \underline{M}^{-1} \underline{L} \underline{D}$$

is Hermitian. But  $\underline{M}^{-1} \underline{L}$  is lower triangular with unit diagonal entries (Properties 7.1 and 7.2). Hence, we must have

$$\underline{M}^{-1} \underline{L} = \underline{I}.$$

### Theorem 7.3

If  $\underline{A}$  is PD, then it is nonsingular and has

$$\det(\underline{A}(\{1, \dots, k\})) > 0, \quad \forall k=1, \dots, n$$

(cf. Property 5.1). Hence, from Theorem 7.1 there exists a unique LU decomposition or LDM decomposition for  $\underline{A}$ .

Since  $\underline{A}$  is Hermitian (by definition), we have from Theorem 7.2 that  $\underline{L} = \underline{M}$ . Furthermore,  $\underline{D} = \text{diag}(d_1, \dots, d_n)$  must satisfy  $d_i > 0 \forall i$ , since for any  $\underline{x} \neq 0$ ,

$$\begin{aligned} 0 &< \underline{x}^H \underline{A} \underline{x} \\ &= \underline{x}^H \underline{L} \underline{D} \underline{L}^H \underline{x} \\ &= \sum_{i=1}^n d_i |z_i|^2, \quad \underline{z} = \underline{L}^H \underline{x}. \end{aligned}$$

Let  $\underline{D}^{1/2} = \text{Diag}(\sqrt{d_1}, \dots, \sqrt{d_n})$ . The Cholesky factor is given by

$$\underline{G} = \underline{L} \underline{D}^{1/2}.$$