LECTURE 6

Theorem 6.1:

By Taylor series expansion,

$$\underline{x}(\epsilon) = \underline{x} + \epsilon \dot{\underline{x}}(0) + O(\epsilon^2) \qquad (1)$$

(Recall $\underline{x} = \underline{x}(0) = \underline{A}^{-1}\underline{b}$), where $\dot{\underline{x}}(\epsilon)$ is the derivative of $\underline{x}(\epsilon)$ w.r.t. ϵ. Taking differentiation on $(\underline{A} + \epsilon \underline{F})\underline{x}(\epsilon) = \underline{b} + \epsilon \underline{f}$, we obtain

$$\underline{A}\dot{\underline{x}}(\epsilon) + \underline{F}\underline{x}(\epsilon) + \epsilon \underline{F}\dot{\underline{x}}(\epsilon) = \underline{f} \qquad (2)$$

For $\epsilon = 0$, (2) equals

$$\dot{\underline{x}}(0) = \underline{A}^{-1}(\underline{f} - \underline{F}\underline{x}).$$

It then follows from (1) that

$$\frac{\|\underline{x}(\epsilon) - \underline{x}\|}{\|\underline{x}\|} = \frac{\|\epsilon \dot{\underline{x}}(0) + O(\epsilon^2)\|}{\|\underline{x}\|}$$

$$\leq \frac{\|\epsilon \underline{A}^{-1}(\underline{f} - \underline{F}\underline{x})\|}{\|\underline{x}\|} + O(\epsilon^2)$$

$$\leq |\epsilon| \, \|\underline{A}^{-1}\| \frac{\|\underline{f} - \underline{F}\underline{x}\|}{\|\underline{x}\|}$$

$$\leq |\epsilon| \, \|\underline{A}^{-1}\| \frac{\|\underline{f}\| + \|\underline{F}\| \|\underline{x}\|}{\|\underline{x}\|}$$

$$= |\epsilon| \, \|\underline{A}^{-1}\| \left\{ \frac{\|\underline{f}\|}{\|\underline{x}\|} + \|\underline{F}\| \right\} \qquad (3)$$

$$\cancel{= |\epsilon| \, \|\underline{A}^{-1}\| \left\{ \frac{\|\underline{f}\|}{\|\underline{A}^{-1}\underline{b}\|} + \|\underline{F}\| \right\}}$$

Since

$$\|\underline{b}\| = \|\underline{A}\underline{x}\| \leq \|\underline{A}\|\|\underline{x}\|$$

we have

$$\|\underline{x}\| \geq \frac{\|\underline{b}\|}{\|\underline{A}\|} \tag{4}$$

Substituting (4) into (3),

$$\frac{\|\underline{x}(\epsilon) - \underline{x}\|}{\|\underline{x}\|} \leq |\epsilon| \, \|\underline{A}^{-1}\| \left\{ \frac{\|\underline{f}\| \, \|\underline{A}\|}{\|\underline{b}\|} + \|\underline{E}\| \right\}$$

$$= |\epsilon| \, \|\underline{A}^{-1}\| \, \|\underline{A}\| \left\{ \frac{\|\underline{f}\|}{\|\underline{b}\|} + \frac{\|\underline{E}\|}{\|\underline{A}\|} \right\} .$$

Suppose that $\underline{I}-\underline{F}$ is singular. It follows that $(\underline{I}-\underline{F})\underline{x} = \underline{0}$ for some $\underline{x} \neq \underline{0}$. But then $\cancel{\|\underline{F}\underline{x}\| =} \|\underline{x}\| = \|\underline{F}\underline{x}\| \leq \|\underline{F}\|\|\underline{x}\|$ implies that $\|\underline{F}\| \geq 1$, which is a contradiction.

Consider

$$\sum_{k=0}^{N} \underline{F}^k (\underline{I}-\underline{F}) = \sum_{k=0}^{N} \underline{F}^k - \sum_{k=0}^{N} \underline{F}^{k+1}$$

$$= \underline{F}^0 - \underline{F}^{N+1} = \underline{I} - \underline{F}^{N+1}$$

Since $\|\underline{F}^k\| \leq \|\underline{F}\|^k$, we have $\lim_{k\to\infty} \|\underline{F}^k\| = 0$ implying that $\lim_{k\to\infty} \underline{F}^k = \underline{0}$. Thus,

$$\sum_{k=0}^{\infty} \underline{F}^k (\underline{I}-\underline{F}) = \underline{I}$$

The nonsingularity of $\underline{I}-\underline{F}$ implies that the inverse of $\underline{I}-\underline{F}$ is $\sum_{k=0}^{\infty} \underline{F}^k$.

Finally,

$$\|(\underline{I}-\underline{F})^{-1}\| = \|\sum_{k=0}^{\infty} \underline{F}^k\|$$

$$\leq \sum_{k=0}^{\infty} \|\underline{F}\|^k$$

$$= \frac{1}{1-\|\underline{F}\|} \qquad \left(\text{since } \|\underline{F}\| < 1\right)$$

Theorem 6.2

The matrix $A + E$ can be expressed as

$$A + E = A(I - F)$$

$$F = -A^{-1}E.$$

Note $\|F\| = r$.

By Lemma 6.1, we know $I - F$ is nonsingular. Hence,

$$(A + E)^{-1} = (I - F)^{-1} A^{-1} = \sum_{k=0}^{\infty} F^k A^{-1}$$

and as a result,

$$\begin{aligned}
\| (A + E)^{-1} - A^{-1} \| &= \left\| \left(\sum_{k=0}^{\infty} F^k - I \right) A^{-1} \right\| \\
&= \left\| \left(\sum_{k=1}^{\infty} F^k \right) A^{-1} \right\| \qquad (\because F^0 = I) \\
&\leq \left\| \sum_{k=1}^{\infty} F^k \right\| \|A^{-1}\| \\
&\leq \|A^{-1}\| \left\{ \sum_{k=1}^{\infty} \|F^k\| \right\} \\
&= \|A^{-1}\| \left\{ \sum_{k=1}^{\infty} r^k \right\} \\
&= \|A^{-1}\| \left\{ r \cdot \sum_{k=0}^{\infty} r^k \right\} \\
&= \|A^{-1}\| \left\{ \frac{r}{1 - r} \right\} \\
&= \frac{\|A^{-1}\| \, \|-A^{-1}E\|}{1 - r} \\
&\leq \frac{\|A^{-1}\|^2 \, \|E\|}{1 - r}
\end{aligned}$$

Lemma 6.2:

Since

$$\| A^{-1} \Delta A \| \leq \| A^{-1} \| \, \| \Delta A \|$$
$$\leq \epsilon \, \kappa(A) \quad = r \quad < 1 ,$$

it follows from Theorem 6.1 that $A + \Delta A$ is nonsingular.
The expression

$$(A + \Delta A) y = b + \Delta b$$

can be rewritten as

$$y = (I + A^{-1} \Delta A)^{-1} A^{-1} (b + \Delta b)$$
$$= (I + A^{-1} \Delta A)^{-1} (x + A^{-1} \Delta b)$$

Then,

$$\| y \| \leq \| (I + A^{-1} \Delta A)^{-1} \| \left(\| x \| + \| A^{-1} \| \, \| \Delta b \| \right)$$

Using Lemma 6.1,

$$\| y \| \leq \frac{1}{1 - r} \left(\| x \| + \| A^{-1} \| \, \| \Delta b \| \right)$$
$$\leq \frac{1}{1 - r} \left(\| x \| + \epsilon \| A^{-1} \| \, \| b \| \right)$$
$$\leq \frac{1}{1 - r} \left(\| x \| + \epsilon \| A^{-1} \| \, \| A \| \, \| x \| \right)$$
$$= \frac{1 + r}{1 - r} \| x \| .$$

Theorem 6.3:

The expression $(\underline{A}+\Delta\underline{A})\underline{y} = \underline{b}+\Delta\underline{b}$ can be rewritten as

$$\underline{y}-\underline{x} = \underline{A}^{-1}(\Delta\underline{b} - \Delta\underline{A}\underline{y})$$

Thus,

$$\|\underline{y}-\underline{x}\| \le \|\underline{A}^{-1}\|\left(\|\Delta\underline{b}\| + \|\Delta\underline{A}\|\|\underline{y}\|\right)$$

$$\le \epsilon\|\underline{A}^{-1}\|\left(\|\underline{b}\| + \|\underline{A}\|\|\underline{y}\|\right)$$

$$\le \epsilon\|\underline{A}^{-1}\|\|\underline{A}\|\left(\|\underline{x}\| + \|\underline{y}\|\right)$$

$$\frac{\|\underline{y}-\underline{x}\|}{\|\underline{x}\|} \le \epsilon\kappa(\underline{A})\left(1+\frac{\|\underline{y}\|}{\|\underline{x}\|}\right)$$

By Lemma 6.2,

$$\frac{\|\underline{y}-\underline{x}\|}{\|\underline{x}\|} \le \epsilon\kappa(\underline{A})\left(1+\frac{1+r}{1-r}\right)$$

$$= 2\epsilon\kappa(\underline{A})/1-r.$$