

Theorem 6.1:

By Taylor series expansion,

$$\underline{x}(\epsilon) = \underline{x} + \epsilon \dot{\underline{x}}(0) + O(\epsilon^2) \quad (1)$$

(Recall $\underline{x} = \underline{x}(0) = A^{-1}\underline{b}$), where $\dot{\underline{x}}(\epsilon)$ is the derivative of $\underline{x}(\epsilon)$ w.r.t. ϵ . Taking differentiation on $(A + \epsilon E)\underline{x}(\epsilon) = \underline{b} + \epsilon \underline{f}$, we obtain

$$A\dot{\underline{x}}(\epsilon) + E\underline{x}(\epsilon) + \epsilon E\dot{\underline{x}}(\epsilon) = \underline{f} \quad (2)$$

For $\epsilon = 0$, (2) equals

$$\dot{\underline{x}}(0) = A^{-1}(\underline{f} - E\underline{x}).$$

It then follows from (1) that

$$\begin{aligned} \frac{\|\underline{x}(\epsilon) - \underline{x}\|}{\|\underline{x}\|} &= \frac{\|\epsilon \dot{\underline{x}}(0)\|}{\|\underline{x}\|} + o(\epsilon^2) \\ &\leq \frac{\|\epsilon A^{-1}(\underline{f} - E\underline{x})\|}{\|\underline{x}\|} + o(\epsilon^2) \\ &\leq |\epsilon| \|A^{-1}\| \frac{\|\underline{f} - E\underline{x}\|}{\|\underline{x}\|} \\ &\leq |\epsilon| \|A^{-1}\| \frac{\|\underline{f}\| + \|E\| \|\underline{x}\|}{\|\underline{x}\|} \\ &= |\epsilon| \|A^{-1}\| \left\{ \frac{\|\underline{f}\|}{\|\underline{x}\|} + \|E\| \right\} \quad (3) \\ &= \cancel{|\epsilon| \|A^{-1}\| \left\{ \frac{\|\underline{f}\|}{\|A^{-1}\underline{b}\|} + \|E\| \right\}} \end{aligned}$$

Since

$$\|b\| = \|Ax\| \leq \|A\| \|x\|$$

we have

$$\|x\| \geq \frac{\|b\|}{\|A\|} \tag{4}$$

Substituting (4) into (3),

$$\begin{aligned} \frac{\|x(\epsilon) - x\|}{\|x\|} &\leq |\epsilon| \|A^{-1}\| \left\{ \frac{\|f\| \|A\|}{\|b\|} + \|E\| \right\} \\ &= |\epsilon| \|A^{-1}\| \|A\| \left\{ \frac{\|f\|}{\|b\|} + \frac{\|E\|}{\|A\|} \right\}. \end{aligned}$$

Suppose that $\underline{I} - \underline{F}$ is singular. It follows that $(\underline{I} - \underline{F})\underline{x} = \underline{0}$ for some $\underline{x} \neq \underline{0}$. But then $\|\underline{F}\underline{x}\| = \|\underline{x}\| = \|\underline{F}\underline{x}\| \leq \|\underline{F}\|\|\underline{x}\|$ implies that $\|\underline{F}\| \geq 1$, which is a contradiction.

Consider

$$\begin{aligned} \sum_{k=0}^N \underline{F}^k (\underline{I} - \underline{F}) &= \sum_{k=0}^N \underline{F}^k - \sum_{k=0}^N \underline{F}^{k+1} \\ &= \underline{F}^0 - \underline{F}^{N+1} = \underline{I} - \underline{F}^{N+1} \end{aligned}$$

Since $\|\underline{F}^k\| \leq \|\underline{F}\|^k$, we have $\lim_{k \rightarrow \infty} \|\underline{F}^k\| = 0$ implying that $\lim_{k \rightarrow \infty} \underline{F}^k = \underline{0}$. Thus,

$$\sum_{k=0}^{\infty} \underline{F}^k (\underline{I} - \underline{F}) = \underline{I}$$

The nonsingularity of $\underline{I} - \underline{F}$ implies that the inverse of $\underline{I} - \underline{F}$ is $\sum_{k=0}^{\infty} \underline{F}^k$.

Finally,

$$\begin{aligned} \|(\underline{I} - \underline{F})^{-1}\| &= \left\| \sum_{k=0}^{\infty} \underline{F}^k \right\| \\ &\leq \sum_{k=0}^{\infty} \|\underline{F}\|^k \\ &= \frac{1}{1 - \|\underline{F}\|} \quad (\text{since } \|\underline{F}\| < 1) \end{aligned}$$

Theorem 6.2

The matrix $A + E$ can be expressed as

$$A + E = A(I - F)$$

$$F = -A^{-1}E.$$

Note $\|E\| = r$.

By Lemma 6.1, we know $I - F$ is nonsingular. Hence,

$$\begin{aligned} (A + E)^{-1} &= (I - F)^{-1} A^{-1} \\ &= \sum_{k=0}^{\infty} F^k A^{-1} \end{aligned}$$

and as a result,

$$\begin{aligned} \|(A + E)^{-1} - A^{-1}\| &= \left\| \left(\sum_{k=0}^{\infty} F^k - I \right) A^{-1} \right\| \\ &= \left\| \left(\sum_{k=1}^{\infty} F^k \right) A^{-1} \right\| \quad (\because F^0 = I) \\ &\leq \left\| \sum_{k=1}^{\infty} F^k \right\| \|A^{-1}\| \\ &\leq \|A^{-1}\| \left\{ \sum_{k=1}^{\infty} \|F^k\| \right\} \\ &= \|A^{-1}\| \left\{ \sum_{k=1}^{\infty} r^k \right\} \\ &= \|A^{-1}\| \left\{ r \cdot \sum_{k=0}^{\infty} r^k \right\} \\ &= \|A^{-1}\| \left\{ \frac{r}{1-r} \right\} \\ &= \frac{\|A^{-1}\| \|A^{-1}E\|}{1-r} \\ &\leq \frac{\|A^{-1}\|^2 \|E\|}{1-r} \end{aligned}$$

Since

$$\begin{aligned}\|A^{-1}\Delta A\| &\leq \|A^{-1}\| \|A\Delta\| \\ &\leq \epsilon \in \kappa(A) = \gamma < 1,\end{aligned}$$

it follows from Theorem 6.1 that $A + \Delta A$ is nonsingular.

The expression

$$(A + \Delta A)y = \underline{b} + \Delta \underline{b}$$

can be rewritten as

$$\begin{aligned}y &= (I + A^{-1}\Delta A)^{-1} A^{-1} (\underline{b} + \Delta \underline{b}) \\ &= (I + A^{-1}\Delta A)^{-1} (\underline{x} + A^{-1}\Delta \underline{b})\end{aligned}$$

Then,

$$\|y\| \leq \| (I + A^{-1}\Delta A)^{-1} \| (\|\underline{x}\| + \|A^{-1}\| \|\Delta \underline{b}\|)$$

Using Lemma 6.1,

$$\begin{aligned}\|y\| &\leq \frac{1}{1-\gamma} (\|\underline{x}\| + \|A^{-1}\| \|\Delta \underline{b}\|) \\ &\leq \frac{1}{1-\gamma} (\|\underline{x}\| + \epsilon \|A^{-1}\| \|\underline{b}\|) \\ &\leq \frac{1}{1-\gamma} (\|\underline{x}\| + \epsilon \|A^{-1}\| \|A\| \|\underline{x}\|) \\ &= \frac{1+\gamma}{1-\gamma} \|\underline{x}\|.\end{aligned}$$

Theorem 6.3:

The expression $(A + \Delta A)y = b + \Delta b$ can be rewritten as

$$y - x = A^{-1} (\Delta b - \Delta A y)$$

Thus,

$$\begin{aligned} \|y - x\| &\leq \|A^{-1}\| (\|\Delta b\| + \|\Delta A\| \|y\|) \\ &\leq \epsilon \|A^{-1}\| (\|b\| + \|A\| \|y\|) \\ &\leq \epsilon \|A^{-1}\| \|A\| (\|x\| + \|y\|) \end{aligned}$$

$$\frac{\|y - x\|}{\|x\|} \leq \epsilon \kappa(A) \left(1 + \frac{\|y\|}{\|x\|} \right)$$

By Lemma 6.2,

$$\begin{aligned} \frac{\|y - x\|}{\|x\|} &\leq \epsilon \kappa(A) \left(1 + \frac{1+r}{1-r} \right) \\ &= 2\epsilon \kappa(A) / (1-r) \end{aligned}$$