COM521500
Math. Methods for SP I
Lecture 6: Sensitivity of Square Linear Systems

A square linear system takes the form

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where $\mathbf{A} \in \mathbb{C}^{n \times n}$ is nonsingular.

The objective is to find $\mathbf{x}$, given the data matrices $\mathbf{A}$ and $\mathbf{b}$.

In practice, $\mathbf{A}$ and $\mathbf{b}$ may contain errors; e.g., floating point errors.

In that case we are solving a perturbed system:

$$(\mathbf{A} + \Delta\mathbf{A})\mathbf{y} = \mathbf{b} + \Delta\mathbf{b}$$

where $\mathbf{y}$ is the solution of the above system. The vector $\mathbf{y}$ can be represented by

$$\mathbf{y} = \mathbf{x} + \Delta\mathbf{x}$$

We are interested in studying how $\Delta\mathbf{x}$ is affected by $\Delta\mathbf{A}$ and $\Delta\mathbf{b}$.

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For notational convenience, define $\|.\|$ to be a (arbitrary) p-norm.

Consider the following perturbed system formulation:

$$(\mathbf{A} + \epsilon\mathbf{F})\mathbf{x}(\epsilon) = \mathbf{b} + \epsilon\mathbf{f}$$

where ϵ determines the scale of perturbation, and $\mathbf{F}$ and $\mathbf{f}$ are error matrices with a constant norm.

Note that $\mathbf{x}(0) = \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

Theorem 6.1

$$\frac{\|\mathbf{x}(\epsilon)-\mathbf{x}\|}{\|\mathbf{x}\|} \leq \epsilon\kappa(\mathbf{A})\left(\frac{\|\mathbf{f}\|}{\|\mathbf{b}\|}+\frac{\|\mathbf{F}\|}{\|\mathbf{A}\|}\right)+O(\epsilon^2)$$

where

$$\kappa(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^{-1}\|$$

is called the **condition number**.

Note that if the 2-norm is used, then the cond. number is

$$\kappa_2(\mathbf{A}) = \frac{\sigma_1(\mathbf{A})}{\sigma_n(\mathbf{A})}.$$

Theorem 6.1 shows that fixing ϵ, the error in $\mathbf{x}(\epsilon)$ increases as the condition number increases.

If $\mathbf{A}$ is close to singular, then $\kappa_2(\mathbf{A}) \gg 1$ and there is significant error in $\mathbf{x}(\epsilon)$ even for very small ϵ.

The above sensitivity analysis is accurate for small ϵ, in which case $O(\epsilon^2)$ is negligible.

We look at an alternate analysis that is mathematically more rigorous.

Lemma 6.1 If $\mathbf{F} \in \mathbb{C}^{n\times n}$ and $\|\mathbf{F}\| < 1$, then $\mathbf{I} - \mathbf{F}$ is nonsingular and

$$(\mathbf{I} - \mathbf{F})^{-1} = \sum_{k=0}^{\infty} \mathbf{F}^k$$

with

$$\|(\mathbf{I} - \mathbf{F})^{-1}\| \leq \frac{1}{1 - \|\mathbf{F}\|}$$

Theorem 6.2 Suppose that $\mathbf{A}$ is nonsingular, and consider the inverse of $\mathbf{A} + \mathbf{E}$ for some error matrix $\mathbf{E}$. If $r = \|\mathbf{A}^{-1}\mathbf{E}\| < 1$, then $\mathbf{A} + \mathbf{E}$ is nonsingular and

$$\|(\mathbf{A} + \mathbf{E})^{-1} - \mathbf{A}^{-1}\| \leq \frac{\|\mathbf{E}\|\|\mathbf{A}^{-1}\|^2}{1 - r}$$

Lemma 6.2 Suppose

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$
$$(\mathbf{A} + \Delta\mathbf{A})\mathbf{y} = \mathbf{b} + \Delta\mathbf{b}$$

with $\|\Delta\mathbf{A}\| \le \epsilon\|\mathbf{A}\|$ and $\|\Delta\mathbf{b}\| \le \epsilon\|\mathbf{b}\|$. If $\epsilon\kappa(\mathbf{A}) = r < 1$, then $\mathbf{A} + \Delta\mathbf{A}$ is nonsingular and

$$\frac{\|\mathbf{y}\|}{\|\mathbf{x}\|} \le \frac{1+r}{1-r}$$

Theorem 6.3 If the conditions in Lemma 6.2 hold, then

$$\frac{\|\mathbf{y}-\mathbf{x}\|}{\|\mathbf{x}\|} \le \frac{2\epsilon}{1-r}\kappa(\mathbf{A})$$