LECTURE 5

Property α.1:

Let

$$\underline{x} = \sum_{i=1}^{r} \alpha_i \underline{e}_{k_i}$$

Then, for PSD $\underline{A}$,

$$0 \leq \underline{x}^H \underline{A}\, \underline{x} = \sum_{i=1}^{r} \sum_{l=1}^{r} \alpha_i^* \alpha_l \, a_{k_i, k_l}$$

$$= \underline{\alpha}^H \underline{A}(\{k_1, \ldots, k_r\})\, \underline{\alpha}$$

thus implying that $\underline{A}(\{k_1, \ldots, k_r\})$ is PSD.

Property α.2:

The quadratic form

$$\underline{x}^H \underline{C}^H \underline{A}\, \underline{C}\, \underline{x}$$

can be rewritten as

$$\underline{z}^H \underline{A}\, \underline{z}$$

where $\underline{z} = \underline{C}\,\underline{x}$. Hence, if $\underline{z}^H \underline{A}\, \underline{z} \stackrel{\geq}{>} 0$ then $\underline{x}^H \underline{C}^H \underline{A}\, \underline{C}\, \underline{x} \geq 0$, implying that $\underline{C}^H \underline{A}\, \underline{C}$ is PSD. Moreover, in order to have $\underline{x}^H \underline{C}^H \underline{A}\, \underline{C}\, \underline{x} = 0$ for some $\underline{x} \neq \underline{0}$, we need

$$\underline{z} = \underline{C}\,\underline{x} = \underline{0}$$

for some $\underline{x} \neq \underline{0}$. For $\text{rank}(\underline{C}) = m$ or $N(\underline{C}) = \{\underline{0}\}$, such a possibility does not occur.

Theorem 4.1:

By eigendecomposition $\underline{A} = \underline{V}\underline{\Lambda}\underline{V}^H$,

$$\underline{x}^H \underline{A} \underline{x} = \underline{z}^H \underline{\Lambda} \underline{z} = \sum_{i=1}^{n} \lambda_i |z_i|^2$$

where $\underline{z} = \underline{V}^H \underline{x}$. The above eqn. is non-negative for any $\underline{z} \neq \underline{0}$ iff $\lambda_i \geq 0 \ \forall \ i$. Likewise the above eqn. is positive for any $\underline{z} \neq \underline{0}$ iff $\lambda_i > 0 \ \forall \ i$.

Theorem 4.2:

Necessity ($\Rightarrow$):

If $\underline{A} = \underline{B}^H \underline{B}$, then

$$\underline{x}^H \underline{A} \underline{x} = \underline{x}^H \underline{B}^H \underline{B} \underline{x} = \| \underline{B} \underline{x} \|_2^2 \geq 0.$$

and hence $\underline{A}$ is PSD.

Sufficiency ($\Leftarrow$):

If $\underline{A}$ is PD or PSD, we can have the following decomposition

$$\underline{A} = \underline{V}\underline{\Lambda}\underline{V}^H = \underline{V}\underline{\Lambda}^{1/2}\underline{\Lambda}^{1/2}\underline{V}^H = (\underline{\Lambda}^{1/2}\underline{V}^H)^H(\underline{\Lambda}^{1/2}\underline{V}^H).$$

which shows that $\underline{A}$ is square-root factorizable.