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# COM521500

## Math. Methods for SP I

### Lecture 5: Positive Semidefinite Matrices

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COM521500 Math. Methods for Signal Processing I

Lecture 5: Positive Semidefinite Matrices

## Quadratic Form

The quantity

$$\mathbf{x}^H \mathbf{A} \mathbf{x}$$

is called the **quadratic form**. It can be expressed as

$$\mathbf{x}^H \mathbf{A} \mathbf{x} = \sum_{i=1}^n \sum_{k=1}^n a_{ik} x_i^* x_k$$

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Consider complex-valued, Hermitian  $\mathbf{A}$ .

The quadratic form  $\mathbf{x}^H \mathbf{A} \mathbf{x}$  is real-valued for any  $\mathbf{x} \in \mathbb{C}^n$ .

Consider real-valued  $\mathbf{A}$ . Every  $\mathbf{A} \in \mathbb{R}^{n \times n}$  can be written as

$$\mathbf{A} = \mathbf{T} + \mathbf{S}$$

where  $\mathbf{T} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$  is symmetric, and  $\mathbf{S} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$  is skew-symmetric; i.e.,  $\mathbf{S}^T = -\mathbf{S}$ .

It can be verified that  $\mathbf{x}^T \mathbf{S} \mathbf{x} = 0$  for any  $\mathbf{x} \in \mathbb{R}^n$ . Hence,  $\mathbf{x}^T \mathbf{A} \mathbf{x}$  only depends on the symmetric part of  $\mathbf{A}$ .

Every  $\mathbf{A} \in \mathbb{R}^{n \times n}$  can be written as

$$\mathbf{A} = \mathbf{T} + \mathbf{S}$$

where

$$\mathbf{T} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

is symmetric, and

$$\mathbf{S} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$$

is skew-symmetric; i.e.,  $\mathbf{S}^T = -\mathbf{S}$ .

It can be verified that  $\mathbf{x}^T \mathbf{S} \mathbf{x} = 0$  for any  $\mathbf{x}$ .

## Positive Definite/Semidefinite Matrices

A Hermitian matrix  $\mathbf{A} \in \mathbb{C}^n$  is said to be **positive semidefinite (PSD)** if

$$\mathbf{x}^H \mathbf{A} \mathbf{x} \geq 0$$

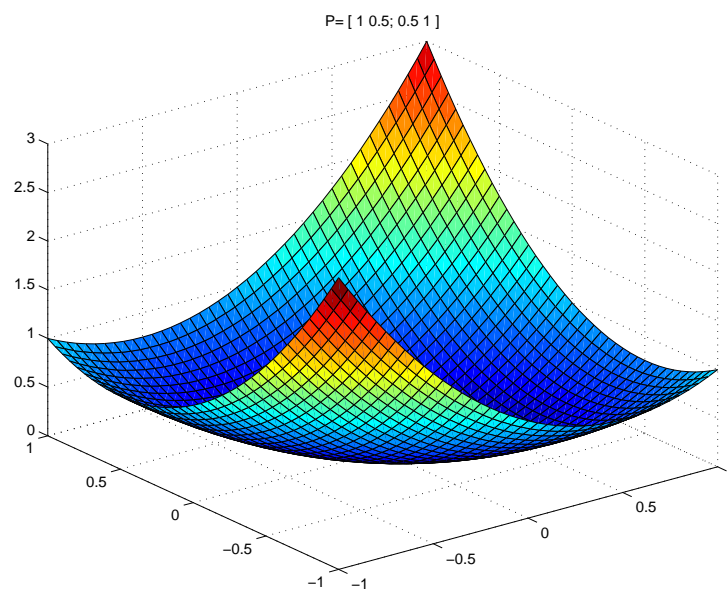
for any  $\mathbf{x} \in \mathbb{C}^n$ ,  $\mathbf{x} \neq \mathbf{0}$ .

A Hermitian matrix  $\mathbf{A} \in \mathbb{C}^n$  is said to be **positive definite (PD)** if

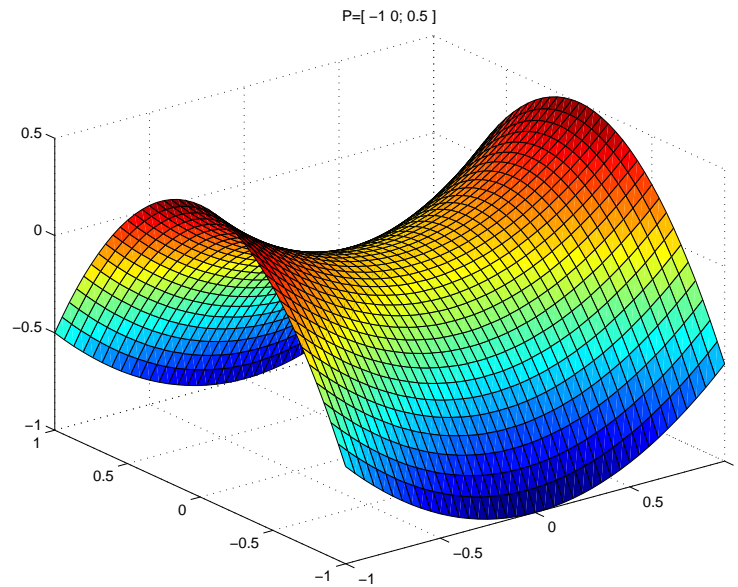
$$\mathbf{x}^H \mathbf{A} \mathbf{x} > 0$$

for any  $\mathbf{x} \in \mathbb{C}^n$ ,  $\mathbf{x} \neq \mathbf{0}$ .

A Hermitian matrix that is not PD or PSD is called an **indefinite** matrix.



Quadratic form for a positive definite matrix.



Quadratic form for an indefinite matrix.

PD and PSD matrices are frequently encountered in practice.

For example, the covariance matrix for a random process  $\mathbf{x}[n] \in \mathbb{C}^N$

$$\begin{aligned} \mathbf{R}_x &= \mathbb{E}\{\mathbf{x}[n]\mathbf{x}^H[n]\} \\ &= \begin{bmatrix} \mathbb{E}\{|x_1[n]|^2\} & \dots & \mathbb{E}\{x_1[n]x_N^*[n]\} \\ \vdots & \ddots & \vdots \\ \mathbb{E}\{x_N[n]x_1^*[n]\} & \dots & \mathbb{E}\{x_N[n]x_N^*[n]\} \end{bmatrix} \end{aligned}$$

is always PSD.

A **principal submatrix** of  $\mathbf{A} \in \mathbb{C}^N$ , denoted by

$$\mathbf{A}(\{k_1, \dots, k_r\})$$

where  $\{k_1, \dots, k_r\} \subset \{1, 2, \dots, n\}$  is an index set, is a matrix obtained by keeping only the  $k_i$ th rows and columns of  $\mathbf{A}$  for  $i = 1, 2, \dots, r$ .

**Property 4.1** If  $\mathbf{A}$  is PSD, then any principal submatrix is PSD.

**Example:** Partition

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

If  $\mathbf{A}$  is PSD, then  $\mathbf{A}_{11}$  and  $\mathbf{A}_{22}$  are PSD.

**Property 4.2** Let  $\mathbf{A} \in \mathbb{C}^n$  be PD. If  $\mathbf{C} \in \mathbb{C}^{n \times m}$  then  $\mathbf{C}^H \mathbf{A} \mathbf{C}$  is PSD. Furthermore,  $\mathbf{C} \in \mathbb{C}^{n \times m}$  is PD if and only if  $\text{rank}(\mathbf{C}) = m$ .

**Example:** Let  $\mathbf{x}[n]$  be a WSS process with covariance  $\mathbf{R}_x$ , and consider another process

$$\mathbf{y}[n] = \mathbf{C}^H \mathbf{x}[n]$$

The covariance of  $\mathbf{y}[n]$  is

$$\mathbf{R}_y = \mathbf{C}^H \mathbf{R}_x \mathbf{C}$$

which is PSD.

**Theorem 4.1** A Hermitian matrix  $\mathbf{A}$  is PSD if and only if all the eigenvalues of  $\mathbf{A}$  are non-negative. A Hermitian matrix  $\mathbf{A}$  is PD if and only if all the eigenvalues of  $\mathbf{A}$  are positive.

It follows that

1. PD matrices are always invertible;
2.  $\text{tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$  is positive/non-negative for a PD/PSD matrix;
3.  $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$  is positive/non-negative for a PD/PSD matrix.

**Theorem 4.2** A Hermitian matrix  $\mathbf{A}$  can be decomposed into the form

$$\mathbf{A} = \mathbf{B}^H \mathbf{B} \quad (*)$$

if and only if  $\mathbf{A}$  is PD or PSD.

The matrix  $\mathbf{B}$  in (\*) is called a square root factor. It is not unique.