Property 3.1

A matrix having a structure of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ z_1^{-1} & z_2^{-1} & & z_L^{-1} \\ z_1^{-2} & z_2^{-2} & & z_L^{-2} \\ \vdots & \vdots & & \vdots \\ z_1^{-(M-1)} & z_2^{-(M-1)} & \dots & z_L^{-(M-1)} \end{bmatrix} \tag{1}$$

is called a *Vandermonde* matrix. The matrix $\mathbf{A}$ defined in our notes is indeed a special case of Vandermonde matrix where $z_l = e^{j\phi(\theta_l)}$.

Now we show that if $M \geq L$ and $z_k \neq z_l$ for $k \neq l$, then $\mathbf{A}$ is of full rank. Let $\mathbf{a}_k = [\ z_1^{-(k-1)}, z_2^{-(k-1)}, \dots, z_M^{-(k-1)}\ ]^T$ be a vector representing the kth row of $\mathbf{A}$. We are interested in whether the first L rows of $\mathbf{A}$ are linear dependent; i.e.,

$$\sum_{k=1}^{L} \alpha_k \mathbf{a}_k = \mathbf{0} \tag{2}$$

for some $\boldsymbol{\alpha} = [\ \alpha_1, \dots, \alpha_L\ ]^T \in \mathbb{C}^L \neq \mathbf{0}$. Eq. (2) is equivalent to

$$\alpha_1 + \alpha_2 z_l^{-1} + \alpha_3 z_l^{-2} + \dots + \alpha_L z_l^{-(L-1)} = 0, \qquad \text{for all } l = 1, \dots, L \tag{3}$$

Condition (3) implies that there exist an $\boldsymbol{\alpha} \neq \mathbf{0}$ such that the polynomial

$$\alpha_1 + \alpha_2 z^{-1} + \alpha_3 z^{-2} + \dots + \alpha_L z^{-(L-1)} \tag{4}$$

has L distinct roots given by $\{z_l\}_{l=1}^{L}$. However, (4) is a polynomial of order $L-1$ and has $L-1$ roots only. In other words, the linear dependence condition in (2) does not hold and thus $\{\mathbf{a}_k\}_{k=1}^{L}$ are linearly independent.

The linear independence of $\{\mathbf{a}_k\}_{k=1}^{L}$ implies that $\text{rank}(\mathbf{A}) \geq L$. As $\text{rank}(\mathbf{A}) \leq \min(L, M) = L$, we have $\text{rank}(\mathbf{A}) = L$ meaning that $\mathbf{A}$ is of full rank.

Property 3.2

To prove Property 2 we consider the dimension of the range space of $\mathbf{A}\mathbf{R}_\mathbf{s}\mathbf{A}^H$. The range space of $\mathbf{A}\mathbf{R}_s\mathbf{A}^H$ is expressed as

$$\mathcal{R}(\mathbf{A}\mathbf{R}_s\mathbf{A}^H) = \{\mathbf{y} \in \mathbb{C}^P \mid \mathbf{y} = \mathbf{A}\mathbf{R}_\mathbf{s}\mathbf{A}^H\mathbf{x}, \quad \mathbf{x} \in \mathbb{C}^P\} \tag{5}$$

$$= \{\mathbf{y} \in \mathbb{C}^P \mid \mathbf{y} = \mathbf{A}\mathbf{R}_s\mathbf{z}_1, \quad \mathbf{z}_1 \in \mathcal{R}(\mathbf{A}^H)\} \tag{6}$$

Since $\dim\{\mathcal{R}(\mathbf{A}^H)\} = \text{rank}(\mathbf{A}^H) = K$, $\mathcal{R}(\mathbf{A}^H)$ covers all elements in $\mathbb{C}^K$; i.e., $\mathcal{R}(\mathbf{A}^H) = \mathbb{C}^K$. Thus,

$$\mathcal{R}(\mathbf{A}\mathbf{R}_s\mathbf{A}^H) = \{\mathbf{y} \in \mathbb{C}^P \mid \mathbf{y} = \mathbf{A}\mathbf{R}_s\mathbf{z}_1, \quad \mathbf{z}_1 \in \mathbb{C}^K\} \tag{7}$$

$$= \{\mathbf{y} \in \mathbb{C}^P \mid \mathbf{y} = \mathbf{A}\mathbf{z}_2, \quad \mathbf{z}_2 \in \mathcal{R}(\mathbf{R}_\mathbf{s})\} \tag{8}$$

Again, the full rank condition of $\mathbf{R}_s$ implies $\mathcal{R}(\mathbf{R}_s) = \mathbb{C}^K$. Therefore,

$$\mathcal{R}(\mathbf{A}\mathbf{R}_s\mathbf{A}^H) = \mathcal{R}(\mathbf{A}) \tag{9}$$

and subsequently $\dim\{\mathcal{R}(\mathbf{A}\mathbf{R}_s\mathbf{A}^H)\} = \dim\{\mathcal{R}(\mathbf{A})\} = K$.

Property 3.3

If $\{\lambda, \mathbf{v}\}$ is an eigen-pair for $\mathbf{A}\mathbf{R}_s\mathbf{A}^H$, then

$$\begin{aligned} \mathbf{R}_y\mathbf{v} &= (\mathbf{A}\mathbf{R}_s\mathbf{A}^H + \sigma_\nu^2\mathbf{I})\mathbf{v} \\ &= (\lambda + \sigma_\nu^2)\mathbf{v} \end{aligned} \tag{10}$$

forms an eigen-equation for $\mathbf{R}_y$.

Property 3.4

Sufficiency: The eigenvectors associated with the zero eigenvalues satisfy

$$\mathbf{A}\mathbf{R}_s\mathbf{A}^H\mathbf{v}_i = \mathbf{0}, \qquad i = K+1, \ldots, P \tag{11}$$

We rewrite (11) as

$$\mathbf{A}\mathbf{z}_i = \mathbf{0}, \qquad i = K+1, \ldots, P \tag{12}$$

$$\mathbf{z}_i = \mathbf{R}_s\mathbf{A}^H\mathbf{v}_i \tag{13}$$

Since the columns of $\mathbf{A}$ are linearly independent, (12) holds if and only if $\mathbf{z}_i = \mathbf{0}$, or, equivalently,

$$\mathbf{R}_s\mathbf{A}^H\mathbf{v}_i = \mathbf{0} \tag{14}$$

Since $\mathbf{R}_s$ is invertible, the only possibility that (14) holds is $\mathbf{A}^H\mathbf{v}_i = \mathbf{0}$.

Necessity: Suppose that $\mathbf{V}_2^H\mathbf{a}(\theta) = \mathbf{0}$ for some $\theta \neq \theta_i$ for all $i = 1, \ldots, K$. This implies $\mathbf{a}(\theta) \in R_\perp(\mathbf{V}_2) = R(\mathbf{V}_1) = R(\mathbf{A})$ (why?). Then,

$$\mathbf{a}(\theta) = \sum_{k=1}^{K} c_k\mathbf{a}(\theta_k) \tag{15}$$

which is equivalent to that

$$[\ \mathbf{a}(\theta), \mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_K)\ ] \in \mathbb{C}^{P\times(K+1)} \tag{16}$$

is linear dependent. But for $P > K$ Eq. (16) is a Vandemonde matrix with full rank. This contradicts the assumption that $\mathbf{V}_2^H\mathbf{a}(\theta) = \mathbf{0}$.