COM521500
Math. Methods for SP I
Lecture 3: Applications of Eigendecomposition

Karhunen-Loeve Expansion

A quick review of random processes:

Consider a sequence of random signals $\{x_1, x_2, x_3, \ldots\}$. Let

$$r(n, \ell) = \mathrm{E}\{x_n x_\ell^*\}$$

denote the auto-correlation function.

A random process is said to be **wide-sense stationary** (WSS) if

$$r_x(n, \ell) = r_x(n + i, \ell + i)$$

for any i.

The same concepts apply to a vector sequence $\{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$.

Let $\{\mathbf{x}_n\}_{n=1}^{\infty} \in \mathbb{C}^N$ be a sequence of random vector signals.

The signal $\mathbf{x}_n$ is assumed to be WSS with zero mean and covariance

$$\mathrm{E}\{\mathbf{x}_n \mathbf{x}_n^H\} = \mathbf{R}_x$$

Some properties of $\mathbf{R}_x$:

1. $\mathbf{R}_x$ is Hermitian (and sym. for $\mathbf{x}_k \in \mathbb{R}^N$)
2. $\mathbf{R}_x$ is **positive semidefinite** (will be discussed in this course).

Consider an orthonormal expansion of $\mathbf{x}_n$:

$$\mathbf{x}_n = \sum_{i=1}^{n} a_{in} \mathbf{q}_i$$

which can be expressed in a more compact form:

$$\mathbf{x}_n = \mathbf{Q}\mathbf{a}_n$$

Since $\mathbf{Q}$ is unitary,

$$\mathbf{a}_n = \mathbf{Q}^H \mathbf{x}_n$$

Signal representation by orthonormal expansion is very common in SP; e.g., the discrete Fourier transform, and the discrete cosine transform.

Example: discrete Fourier transform

$$\mathbf{Q} = \left[\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N\right], \quad \mathbf{q}_k = \begin{bmatrix} 1 \\ e^{j2\pi k/N} \\ \vdots \\ e^{j2\pi k(N-1)/N} \end{bmatrix}$$

In applications such as coding and compression, both the transmitter and receiver know $\mathbf{Q}$.

The transmitter sends $\mathbf{a}_n$.

At the receiver, $\mathbf{x}_n$ is constructed from $\mathbf{a}_n$.

We are interested in finding a $\mathbf{Q}$ such that the coefficients a_{in} are uncorrelated, thereby eliminating redundancy.

Let's take a look at the covariance matrix of $\mathbf{a}_n$:

$$\begin{aligned}\mathbf{R}_a &= \mathrm{E}\{\mathbf{a}_n\mathbf{a}_n^H\} \\ &= \mathrm{E}\{\mathbf{Q}^H\mathbf{x}_n\mathbf{x}_n^H\mathbf{Q}\} \\ &= \mathbf{Q}^H\mathrm{E}\{\mathbf{x}_n\mathbf{x}_n^H\}\mathbf{Q} \\ &= \mathbf{Q}^H\mathbf{R}_x\mathbf{Q}\end{aligned}$$

Consider the eignedecomposition $\mathbf{R}_x = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$.

Apparently, $\mathbf{R}_a$ is diagonal if (and only if) $\mathbf{Q} = \mathbf{V}$.

The expansion of $\mathbf{x}_n$ using the eigenvectors of its covariance $\mathbf{R}_x$ is called the **Karhunen-Loeve expansion**.

With the Karhunen-Loeve (KL) expansion,

$$\mathbf{R}_a = \begin{bmatrix} \mathrm{E}\{|a_{1n}|^2\} & & & 0 \\ & \mathrm{E}\{|a_{2n}|^2\} & & \\ & & \ddots & \\ 0 & & & \mathrm{E}\{|a_{N,n}|^2\} \end{bmatrix} = \mathbf{\Lambda}$$

Hence, $\lambda_i = \mathrm{E}\{|a_{in}|^2\}$ meaning that the eigenvalues are the average energies of the KL coefficients.

There are many situations where the energy in the first few KL coefficients a_{in} dominates that in the remaining ones.

For convenience, assume $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$.

In coding and compression applications, we consider transmitting only part of the KL coefficients, specifically those that have principal eigenvalues (or average energies):

$$\hat{\mathbf{a}}_n = [\ a_{1n}, a_{2n}, \ldots, a_{r,n}]^T$$

The reconstruction of $\mathbf{x}_n$ (which is an approximation unless $\lambda_{r+1} = \ldots = \lambda_N = 0$) is then done by

$$\hat{\mathbf{x}}_n = \sum_{i=1}^{r} a_{in}\mathbf{v}_i$$

Some final remarks:

1. The KL transform requires knowledge of $\mathbf{R}_x$. In practice we can only estimate it by averaging:

$$\hat{\mathbf{R}}_x = \frac{1}{M}\sum_{n=1}^{M} \mathbf{x}_n\mathbf{x}_n^H$$

   for some window length M.

2. We also need to transmit the eigenvector matrix of $\mathbf{R}_x$, which is not always bandwidth efficient.

3. For a class of covariance models, it has been shown that the discrete cosine transform forms the KL. Thus, we don't need to transmit the eigenvector matrix.

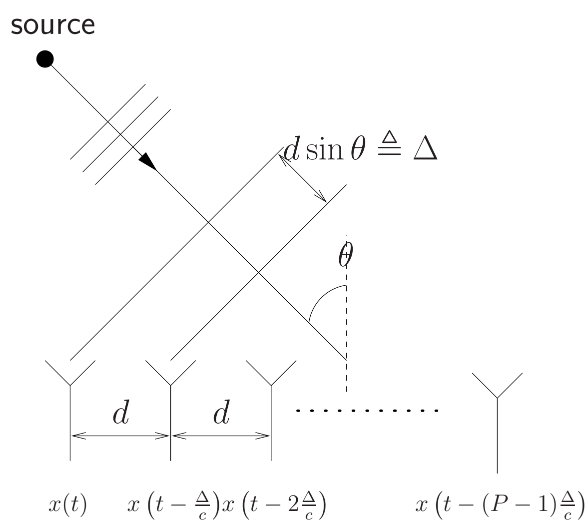
Subspace Methods for Sensor Array Processing

Applications of sensor array processing: radar, sonar, communications, seismology, audio & speech processing, . . .

Two important problems in sensor array processing:

- *Source Localization*: estimate the source locations; e.g., the (x, y, z) coordinate, and the direction of arrival (DOA).
- *Beamforming*: extract the desired source signal from the received signals, given that the source location.

We are interested in **DOA estimation in uniform linear arrays**.



Uniform linear array

Assume far-field situations in which cases source waves are planar.

Supposing that there is only one radiating source in the free space, the output of sensor p can be represented by

$$\tilde{y}_p(t) = x\left(t - (p-1)\frac{d\sin\theta}{c}\right)$$

where

$x(t)$ represents the source signal impinging on sensor 1,

θ is the DOA of the source signal, and

c is the wave propagation velocity.

In many applications, source signals are carrier-modulated:

$$x(t) = e^{j\omega_c t}s(t)$$

Let $y_p(t) = e^{-j\omega_c t}\tilde{y}_p(t)$ be a demodulated signal for sensor p. Then,

$$\begin{aligned} y_p(t) &= e^{-j\omega_c t}x(t-(p-1)d\sin\theta/c) \\ &= e^{-j(p-1)\omega_c d\sin\theta/c}s\left(t-(p-1)d\sin\theta/c\right) \end{aligned}$$

Source signals are called *narrowband* if

$$s\left(t-(p-1)d\sin\theta/c\right) \simeq s(t), \qquad \forall p \in \{1,\ldots,P\}$$

Source signals are called *wideband* if the above assumption does not hold.

Let $\mathbf{y}(t) = [\ y_1(t), \ldots, y_P(t)\ ]^T$. It can be represented by

$$\mathbf{y}(t) = \mathbf{a}(\theta)s(t)$$

Here,

$$\mathbf{a}(\theta) = [\ 1, e^{-j\phi(\theta)}, e^{-2j\phi(\theta)}, \ldots, e^{-j(P-1)\phi(\theta)}\ ]^T,$$

is referred to as a *steering vector*, and

$$\phi(\theta) = \omega_c d \sin\theta / c = 2\pi d \sin\theta / \lambda.$$

where λ is the wavelength of the carrier frequency ω_c.

To avoid spatial aliasing (i.e., $\mathbf{a}(\theta_1) = \mathbf{a}(\theta_2)$ for some $\theta_1 \neq \theta_2$, $\theta_1, \theta_2 \in [\frac{-\pi}{2}, \frac{\pi}{2}]$), we need

$$d \leq \frac{\lambda}{2}$$

Define $\mathbf{y}[n] = \mathbf{y}(nT_s)$ to be a time-sampled version of $\mathbf{y}(t)$.

Multiple signal model:

$$\mathbf{y}[n] = \sum_{k=1}^{K} \mathbf{a}(\theta_k) s_k[n] + \boldsymbol{\nu}[n]$$
$$= \mathbf{A}\mathbf{s}[n] + \boldsymbol{\nu}[n]$$

where $\mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_K)]$, & $\mathbf{s}[n] = [s_1[n], \ldots, s_K[n]]^T$. Here,

$s_k[n]$ is kth source signal,

θ_k is the DOA of the kth source,

$\boldsymbol{\nu}[n]$ is additive spatially white noise.

Assume that $s_k[n]$ are wide-sense stationary.

Consider a correlation matrix $\mathbf{R}_y = \mathrm{E}\{\mathbf{y}[n]\mathbf{y}^H[n]\} \in \mathbb{C}^{P \times P}$.

$$\mathbf{R}_y = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \sigma_\nu^2 \mathbf{I}$$

where $\mathbf{R}_s = \mathrm{E}\{\mathbf{s}[n]\mathbf{s}^H[n]\} \in \mathbb{C}^{K \times K}$.

Assume that

i) $P > K$,

ii) the DOAs θ_k are distinct; and

iii) $s_k[n]$ are not coherent (but can be correlated) to one other such that $\mathbf{R}_s$ is of full rank.

Property 3.1 For distinct θ_k, $\mathbf{A}$ is of full rank.

Consider the eigendecomposition of the signal correlation matrix:

$$\mathbf{A}\mathbf{R}_s\mathbf{A}^H = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$$

where $\mathbf{V} = [\ \mathbf{v}_1, \ldots, \mathbf{v}_P\ ]$, and $\mathbf{\Lambda} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_P)$. We assume $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_P$.

Property 3.2 The number of nonzero eigenvalues of $\mathbf{A}\mathbf{R}_s\mathbf{A}^H$ is K. Or, $\lambda_{K+1} = \ldots = \lambda_P = 0$.

Property 3.3 The eigendecomposition of $\mathbf{R}_x$ is

$$\mathbf{R}_y = \mathbf{V}(\mathbf{\Lambda} + \sigma_\nu^2\mathbf{I})\mathbf{V}^H.$$

Property 3.3 means that the eigenvector matrix of the signal correlation matrix is the same as that of $\mathbf{R}_y$.

Property 3.4 Partition $\mathbf{V} = [\ \mathbf{V}_1\ \mathbf{V}_2\ ]$ where $\mathbf{V}_1 = [\ \mathbf{v}_1, \ldots, \mathbf{v}_K\ ]$ & $\mathbf{V}_2 = [\ \mathbf{v}_{K+1}, \ldots, \mathbf{v}_P\ ]$. We have

$$\mathbf{V}_2^H\mathbf{a}(\theta) = \mathbf{0}$$

if and only if $\theta = \theta_i$ for any $i = 1, \ldots, K$.

MUSIC: MUltiple SIgnal Classification

MUSIC is one of the most well known subspace DOA estimation algorithms.

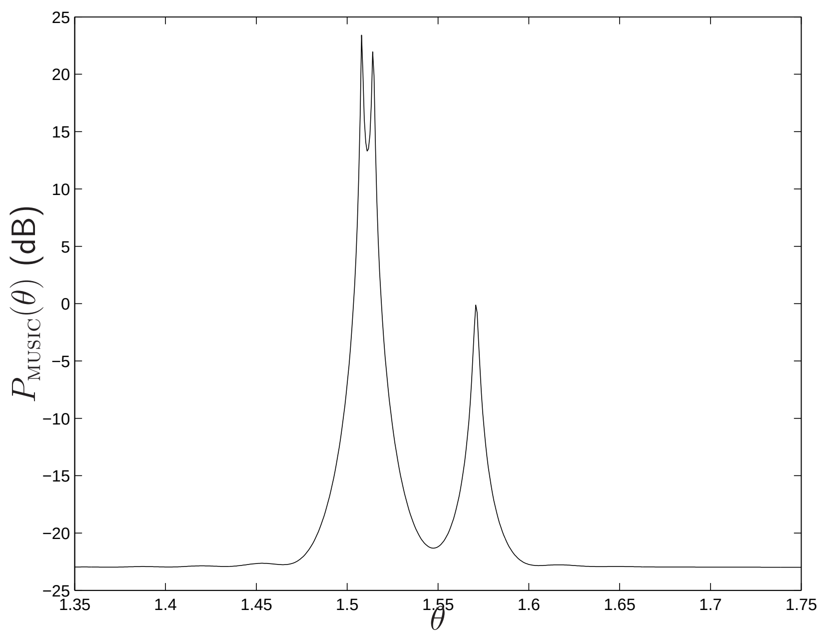
- **Step 1**. Compute the sample correlation matrix

$$\hat{\mathbf{R}}_y = \frac{1}{N}\sum_{n=1}^{N} \mathbf{y}[n]\mathbf{y}^H[n]$$

- **Step 2**. Find the eigenvector matrix of $\hat{\mathbf{R}}_y$, denoted by $\hat{\mathbf{V}}$.

- **Step 3**. Determine the DOAs by finding the peaks of the 'pseudo-spectrum'

$$P_{music}(\theta) = \frac{1}{\|\hat{\mathbf{V}}_2^H \mathbf{a}(\theta)\|_2^2}$$



MUSIC pseudo-spectrum.

Circulant Matrix, & OFDM

A matrix having a structure of

$$\mathbf{H} = \begin{bmatrix} h_0 & h_{N-1} & \dots & h_2 & h_1 \\ h_1 & h_0 & \dots & h_3 & h_2 \\ \vdots & & \ddots & & \vdots \\ h_{N-1} & h_{N-2} & \dots & h_1 & h_0 \end{bmatrix}$$

is called a **circulant matrix**.

Let

$$\mathbf{f}_k = \frac{1}{\sqrt{N}}[\ 1\ e^{j2\pi k/N}\ e^{j4\pi k/N}\ \dots\ e^{j2\pi(N-1)k/N}\ ]^T$$

for $k = 0, 1, \dots, N-1$. It can be verified that

$$\mathbf{H}\mathbf{f}_k = H(e^{j2\pi k/N})\mathbf{f}_k$$

where

$$H(z) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} h_n z^{-n}$$

is a normalized z-transform of $\{h_n\}$.

This means that $\mathbf{f}_k$ is an eigenvector of $\mathbf{H}$, and that $H(e^{j2\pi k/N})$ is an eigenvalue.

Let $\mathbf{F} = [\ \mathbf{f}_0\ \mathbf{f}_1\ \ \ldots\ \ \mathbf{f}_{N-1}\ ]$.

The matrix $\mathbf{F}$ is the inverse discrete Fourier transform (DFT) matrix, and is unitary.

The matrix $\mathbf{F}^{-1} = \mathbf{F}^H$ is the DFT matrix.

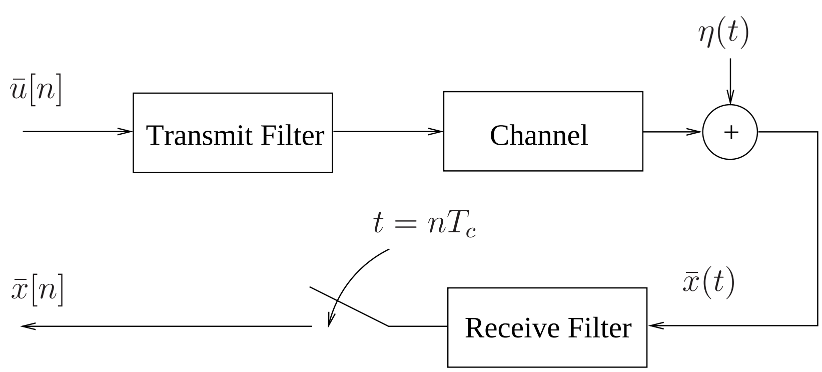
Therefore, $\mathbf{H}$ has an eigendecomposition

$$\mathbf{H} = \mathbf{F}\mathbf{D}\mathbf{F}^H$$

where

$$\mathbf{D} = \mathrm{Diag}(H(e^{j0}), H(e^{j2\pi/N}), \ldots, H(e^{j2\pi(N-1)/N}))$$

Digital communications over linear time-invariant channels



Continuous-time received signal model:

$$\bar{x}(t) = \sum_{n=-\infty}^{\infty} \bar{u}[n]h(t - nT_c) + \bar{\nu}(t)$$

Here,

- $\bar{u}[n]$ transmitted signal sequence
- $h(t)$ overall impulse response of the transmit filter, channel, and receive filter.
- $\bar{\nu}(t)$ noise.

Discrete-time received signal model:

$$\begin{aligned}\bar{x}[n] &= x(t)\big|_{t=nT_c} \\ &= \sum_{\ell=0}^{L} h[\ell]\bar{u}[n-\ell] + \bar{\nu}[n]\end{aligned}$$

where $h[n] = h(t)|_{t=nT_c}$, & $\bar{\nu}[n] = \bar{\nu}(t)|_{t=nT_c}$.

The received signal is subject to inter-symbol interference due to the dispersive effects of $h[n]$.

Orthogonal Frequency Division Multiplexing (OFDM)

Let P be a block length. P is chosen such that $P \gg L$.

Let $\bar{\mathbf{x}}_i = [\ x[iP], x[iP+1], \ldots, x[iP+P-1]\ ]^T$.

$$\bar{\mathbf{x}}_i = \mathbf{H}_0 \bar{\mathbf{u}}_i + \mathbf{H}_1 \bar{\mathbf{u}}_{i-1} + \bar{\boldsymbol{\nu}}_i$$

where

$$\mathbf{H}_0 = \begin{bmatrix} h[0] & 0 & 0 & \ldots & & 0 \\ h[1] & h[0] & 0 & \ldots & & 0 \\ \vdots & & \ddots & & & \\ h[L] & & & \ddots & & \\ \vdots & \ddots & & & \ddots & \\ 0 & & h[L] & & & h[0] \end{bmatrix} \in \mathbb{C}^{P \times P}$$

$$\mathbf{H}_1 = \begin{bmatrix} 0 & \dots & 0 & h[L] & \dots & h[1] \\ \vdots & \ddots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & & \ddots & h[L] \\ \vdots & & & \ddots & & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 \end{bmatrix} \in \mathbb{C}^{P\times P}$$

$\mathbf{H}_1$ leads to interblock interference (IBI).

To obtain IBI-free blocks, let

$$\mathbf{R} = [\ \mathbf{0}_{N,L}\ \mathbf{I}_N\ ] \in \mathbb{C}^{N\times P}$$

be a receive matrix where $N = P - L$. Define

$$\mathbf{x}_i = \mathbf{R}\bar{\mathbf{x}}_i$$

The model for $\mathbf{x}_i$ is

$$\mathbf{x}_i = \mathbf{R}\mathbf{H}_0\bar{\mathbf{u}}_i + \boldsymbol{\nu}_i$$

where $\mathbf{R}\mathbf{H}_1 = \mathbf{0}$.

Note that

$$\mathbf{R}\mathbf{H}_0 = \begin{bmatrix} h[L] & \dots & h[1] & h[0] & & & & \\ & h[L] & \dots & h[1] & h[0] & & & \\ & & \ddots & & & \ddots & \ddots & \\ & & & h[L] & & & h[1] & h[0] \end{bmatrix} \in \mathbb{C}^{N \times P}$$

Cyclic prefix insertion

Let

$$\mathbf{T} = \begin{bmatrix} \mathbf{0}_{N,L} \ \mathbf{I}_L \\ \mathbf{I}_N \end{bmatrix} \in \mathbb{C}^{P \times N}$$

a transmit matrix.

The transmitted block $\bar{\mathbf{u}}_i \in \mathbb{C}^P$ is constructed by another signal block $\mathbf{u}_i \in \mathbb{C}^N$, through the process

$$\bar{\mathbf{u}}_i = \mathbf{T}\mathbf{u}_i$$

The received block $\mathbf{x}_i$ can then be expressed as

$$\mathbf{x}_i = \tilde{\mathbf{H}}_0 \mathbf{u}_i + \boldsymbol{\nu}_i$$

The channel matrix $\tilde{\mathbf{H}}_0 = \mathbf{R}\mathbf{H}_0\mathbf{T} \in \mathbb{C}^{N\times N}$ takes the form

$$\tilde{\mathbf{H}}_0 = \begin{bmatrix} h[0] & & & h[L] & \dots & h[1] \\ h[1] & h[0] & & & \ddots & \vdots \\ \vdots & & \ddots & & & h[L] \\ h[L] & & & \ddots & & \\ & \ddots & & & \ddots & \\ & & h[L] & \dots & h[1] & h[0] \end{bmatrix}$$

which is a circulant matrix.

By eigendecomposition of $\tilde{\mathbf{H}}_0$,

$$\mathbf{x}_i = \mathbf{F}\mathbf{D}\mathbf{F}^H\mathbf{u}_i + \boldsymbol{\nu}_i$$

Let $\mathbf{s}_i \in \mathbb{C}^N$ be a block of data symbols. We form $\mathbf{u}_i$ by an inverse DFT process:

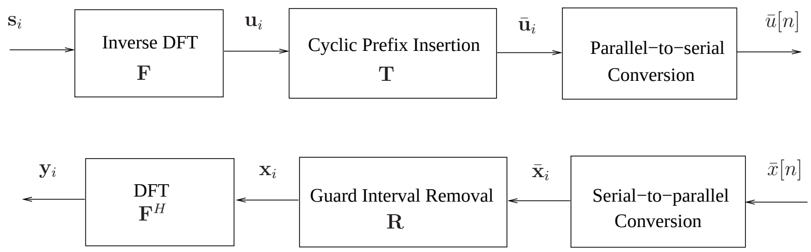
$$\mathbf{u}_i = \mathbf{F}\mathbf{s}_i$$

Let $\mathbf{y}_i = \mathbf{F}^H\mathbf{x}_i$ (i.e., the DFT of $\mathbf{x}_i$). We have

$$\mathbf{y}_i = \mathbf{D}\mathbf{s}_i + \mathbf{F}^H\boldsymbol{\nu}_i$$

where the channel becomes diagonal, thereby achieving zero ISI!

Block transmission processes in OFDM



Additional References

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