Property 2.1:

$$\det(\underline{B} - \lambda \underline{I}) = \det\left(\underline{S}^{-1}(\underline{A} - \lambda \underline{I})\underline{S}\right)$$
$$= \det(\underline{A} - \lambda \underline{I}) \det(\underline{S}^{-1}) \det(\underline{S})$$
$$= \det(\underline{A} - \lambda \underline{I})$$

Property 2.2:

We have from Property 2.1 that the characteristic polynomials for two similar matrices $\underline{A}$ & $\underline{B}$ are the same. Hence $\underline{A}$ and $\underline{B}$ have the same eigenvalues, and determinant.

Theorem 2.1:

Necessity: Suppose that there is a set of linearly independent ~~evector~~ eigenvectors $\{\underline{v}_1, \cdots, \underline{v}_n\}$; i.e.,

$$\underline{A}\underline{v}_i = \lambda_i \underline{v}_i, \quad i = 1, \cdots, n \tag{1}$$

Let $\underline{V} = [\underline{v}_1, \cdots, \underline{v}_n]$, and $\underline{D} = \mathrm{Diag}(\lambda_1, \cdots, \lambda_n)$. Eq. (1) is rewritten as

$$\underline{A}\underline{V} = \underline{V}\underline{D}$$

Since $\underline{V}$ is nonsingular,

$$\underline{V}^{-1}\underline{A}\underline{V} = \underline{D}$$

Sufficiency: Suppose that $\underline{A}$ is diagonalizable so that

$$\underline{S}^{-1}\underline{A}\underline{S} = \underline{D} \tag{2}$$

for some nonsingular $\underline{S}$ and diagonal $\underline{D}$. From Eq. (2),

$$\underline{A}\underline{S} = \underline{S}\underline{D}$$

or

$$\underline{A}\underline{s}_i = d_i \underline{s}_i$$

where $\underline{S} = [\underline{s}_1, \cdots, \underline{s}_n]$, $\underline{D} = \mathrm{Diag}(d_1, \cdots, d_n)$. By the definition of eigenvalues and eigenvectors, ~~s~~ d_i and $\underline{s}_i$ are an eigenvalue and ~~a~~ eigenvector.

Property 2.3

This property is proved by contradiction. Suppose that $\lambda_i \neq \lambda_k$ $\forall$ $i \neq k$, and that $\{\underline{v}_1, \cdots, \underline{v}_k\}$ is linearly dependent. Then there exists $\{c_i\}$ such that

$$\sum_{i=1}^{k} c_i \underline{v}_i = \underline{0} \tag{1}$$

w.l.o.g. assume that $c_1 \neq 0$ and $c_k \neq 0$. Eq. (1) can be rewritten as

$$\sum_{\substack{i=1 \\ c_i \neq 0}}^{k} c_i \underline{v}_i = \underline{0} \tag{2}$$

Premultiplying (2) by $\underline{A}$, we obtain

$$\sum_{\substack{i=1 \\ c_i \neq 0}}^{k} c_i \lambda_i \underline{v}_i = \underline{0} \tag{3}$$

From (2),

$$\lambda_k \sum_{\substack{i=1 \\ c_i \neq 0}}^{k} c_i \underline{v}_i = \underline{0} = \sum_{\substack{i=1 \\ c_i \neq 0}}^{k} c_i \lambda_i \underline{v}_i$$

$$\Leftrightarrow \quad \sum_{\substack{i=1 \\ c_i \neq 0}}^{k-1} c_i (\lambda_k - \lambda_i) \underline{v}_i = \underline{0} \tag{4}$$

Repeating the steps in (3) and (4), we end up with

$$c_1 \prod_{\substack{i=1 \\ c_i \neq 0}}^{k} (\lambda_i - \lambda_1) \underline{v}_1 = \underline{0} \tag{5}$$

Eq. (5) holds only when $\underline{v}_1 = \underline{0}$. This contradicts the definition of eigenvector.

Repeated eigenvalues:

When $\text{rank}(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = n - r$, $\dim N(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = r$. Thus, we can find a set of r linearly independent $\underset{\sim}{v}$ such that

$$(\underset{\sim}{A} - \lambda \underset{\sim}{I}) \underset{\sim}{v} = \underset{\sim}{0}.$$

Property 2.4:

$$\underset{\sim}{A} \underset{\sim}{v} = \lambda \underset{\sim}{v}$$

$$\underset{\sim}{v}^H \underset{\sim}{A} \underset{\sim}{v} = \lambda \underset{\sim}{v}^H \underset{\sim}{v} \qquad (1)$$

LHS of (1):

$$\underset{\sim}{v}^H \underset{\sim}{A} \underset{\sim}{v} = \sum_i \sum_k v_i^* v_k a_{ik}$$

$$= \sum_i |v_i|^2 a_{ii} + \sum_{i>k} v_i^* v_k a_{ik} + \sum_{i<k} v_i^* v_k a_{ik}$$

$$= \sum_i |v_i|^2 a_{ii} + \sum_{i>k} v_i^* v_k a_{ik} + \sum_{i>k} (v_i^* v_k a_{ik})^*$$

$$= \sum_i |v_i|^2 a_{ii} + 2\,\mathrm{Re}\left\{\sum_{i>k} v_i^* v_k a_{ik}\right\} \qquad (2)$$

Note that (2) is due to $a_{ik} = a_{ki}^*$. Since a_{ii} is real for any i, (2) is real. Since the RHS of (1) is $\lambda \|\underset{\sim}{v}\|_2^2$, λ cannot be complex.

Property 2.5:

$$\underline{A}\underline{x}_i = \lambda_i \underline{x}_i$$

$$\underline{x}_k^H \underline{A}\underline{x}_i = \lambda_i \underline{x}_k^H \underline{x}_i$$

$$(\underline{A}^H \underline{x}_k)^H \underline{x}_i = \lambda_i \underline{x}_k^H \underline{x}_i$$

$$(\underline{A}\underline{x}_k)^H \underline{x}_i = \lambda_i \underline{x}_k^H \underline{x}_i$$

$$\lambda_k^* \underline{x}_k^H \underline{x}_i = \lambda_i \underline{x}_k^H \underline{x}_i$$

Since λ_k is real for Hermitian $\underline{A}$ (cf., Property 2.4),

$$(\lambda_k - \lambda_i)\, \underline{x}_k^H \underline{x}_i = \underline{0}.$$

Since $\lambda_k \neq \lambda_i$ for $k \neq i$, $\underline{x}_k^H \underline{x}_i = \underline{0}$.

Theorem 2.2

Consider the following 2 lemmas:

Lemma 2.1: Given a vector $\underline{v}$, there exists a set of vectors $\{y_2, \cdots, y_n\}$ such that $[\underline{v}, y_2, \cdots, y_n]$ is a unitary matrix.

We will see that Lemma 2.1 is true, when we study QR decomposition.

Lemma 2.2 (Matrix deflation) Consider an $n \times n$ matrix $\underline{A}$ where its eigenvalues and eigenvectors are denoted by λ_i and $\underline{v}_i$, respectively. Let $\underline{S}$ be a nonsingular matrix where its first column is $\underline{v}_1$. Then,

$$\underline{S}^{-1}\underline{A}\,\underline{S} = \left[\begin{array}{c|c} \lambda_1 & * \\ \hline \underline{0} & \underline{A}_2 \end{array}\right]. \qquad (1)$$

~~other~~ Moreover, the eigenvalues of $\underline{A}_2$ are $\lambda_2, \cdots, \lambda_n$.

Proof of Lemma 2.2:

$$\underline{A}\underline{v}_1 = \lambda_1 \underline{v}_1$$
$$\underline{S}^{-1}\underline{A}\underline{v}_1 = \lambda_1 \underline{S}^{-1}\underline{v}_1$$
$$\underline{S}^{-1}\underline{A}\,\underline{S}\,\underline{S}^{-1}\underline{v}_1 = \lambda_1 \underline{S}^{-1}\underline{v}_1$$

From the fact $\underline{S}^{-1}\underline{S} = \underline{I}$, we have that $\underline{S}^{-1}\underline{v}_1 = \underline{e}_1$. Hence,

$$\underline{S}^{-1}\underline{A}\,\underline{S}\,\underline{e}_1 = \lambda_1 \underline{e}_1$$

and the matrix structure in (1) follows. Moreover, due to (1),

$$\cancel{\det(\underline{S}^{-1}\underline{A}\,\underline{S}) =}$$
$$\det(\underline{S}^{-1}\underline{A}\,\underline{S} - \lambda\underline{I}) = (\lambda_1 - \lambda)\det(\underline{A}_2 - \lambda\underline{I})$$

Hence, $\det(\underline{A}_2 - \lambda\underline{I}) = \prod_{i=2}^{n}(\lambda_i - \lambda)$ where $\det(\underline{S}^{-1}\underline{A}\,\underline{S} - \lambda\underline{I}) = \prod_{i=1}^{n}(\lambda_i - \lambda)$.

Proof of Theorem 2.2:

Let $\underline{U}_1 = [\, \underline{v}_1, \underline{y}_2, \cdots \underline{y}_n \,]$ where $\underline{v}_1$ is ~~the~~ an eigenvector of $\underline{A}$, and $\{\underline{v}_1, \underline{y}_2, \cdots, \underline{y}_n\}$ is orthonormal (Lemma 2.1). By Lemma 2,

$$\underline{U}_1^H \underline{A}\, \underline{U}_1 = \left[\begin{array}{c|c} \lambda_1 & * \\ \hline \underline{0} & \underline{A}_2 \end{array}\right] \qquad (3)$$

where the eigenvalues of $\underline{A}_2$ are $\lambda_2, \cdots, \lambda_n$. Apply the procedure in (3) on $\underline{A}_2$,

$$\underline{U}_2^H \underline{A}_2\, \underline{U}_2 = \left[\begin{array}{c|c} \lambda_2 & * \\ \hline \underline{0} & \underline{A}_3 \end{array}\right]$$

By letting

$$\underline{U} = \underline{U}_1 \left[\begin{array}{c|c} 1 & \underline{0} \\ \hline \underline{0} & \underline{U}_2 \end{array}\right] \left[\begin{array}{cc|c} 1 & & \underline{0} \\ & 1 & \\ \hline \underline{0} & & \underline{U}_3 \end{array}\right] \cdots \left[\begin{array}{c|c} \underline{I} & \\ \hline & \underline{U}_{n-1} \end{array}\right]$$

we obtain

$$\underline{U}^H \underline{A}\, \underline{U} = \underline{T}$$

Property 2.6: ~~Be~~ Obtain by using the standard result $(\underline{A}\underline{B})^{-1} = \underline{B}^{-1}\underline{A}^{-1}$.

Property 2.7:

Assume that $\lambda_i \neq 0$ for $i = 1, 2, \cdots, r$, and $\lambda_i = 0$ for $i = r+1, \cdots, n$.
The property is proved by considering the dimension of $\mathcal{R}(\underline{A})$:

$$\mathcal{R}(\underline{A}) = \{ \underline{y} \in \mathbb{C}^n \mid \underline{y} = \underline{A}\underline{x},\ \underline{x} \in \mathbb{C}^n \}$$

By eigendecomposition we have

$$\underline{y} = \underline{V}\underline{\Lambda}\underline{V}^H\underline{x}$$

Since $\underline{V}^H$ is a basis for $\mathbb{C}^n$, it is equivalent that

$$\mathcal{R}(\underline{A}) = \{ \underline{y} \in \mathbb{C}^n \mid \underline{y} = \underline{V}\underline{\Lambda}\underline{\alpha},\ \underline{\alpha} \in \mathbb{C}^n \}$$

Now,

$$\begin{aligned} \underline{y} &= \underline{V}\underline{\Lambda}\underline{\alpha} \\ &= \sum_{i=1}^{r} \underline{v}_i \lambda_i \alpha_i \end{aligned}$$

Thus, $\mathcal{R}(\underline{A}) = \text{span}[\underline{v}_1, \cdots, \underline{v}_r]$ and it follows that
$\text{rank}(\underline{A}) = \dim \mathcal{R}(\underline{A}) = r$.

Matrix 2-norm

$$\|A\|_2^2 \equiv \max_{\|x\|_2=1} \|Ax\|_2^2$$

$$= \max_{\|x\|_2=1} x^H A^H A x$$

$A^H A$ is a Hermitian, and hence it has an eigendecomposition

$$A^H A = V \Lambda V^H.$$

Then,

$$x^H A^H A x = x^H V \Lambda V^H x$$

$$= \sum_{i=1}^{n} \lambda_i |v_i^H x|^2$$

$$\leq \sum_{i=1}^{n} \lambda_{max} |v_i^H x|^2$$

$$= \lambda_{max} \|V^H x\|_2^2$$

$$= \lambda_{max} \|x\|_2^2$$

$$= \lambda_{max}$$

, for $\|x\|_2 = 1$

It can be verified that equality holds when $x = v_{max}$. Hence, $\|A\|_2^2 = \lambda_{max}$.