

Property 2.1:

$$\begin{aligned}
\det(\underline{B} - \lambda \underline{I}) &= \det(\underline{S}^{-1}(\underline{A} - \lambda \underline{I})\underline{S}) \\
&= \det(\underline{A} - \lambda \underline{I}) \det(\underline{S}^{-1}) \det(\underline{S}) \\
&= \det(\underline{A} - \lambda \underline{I})
\end{aligned}$$

Property 2.2:

We have from Property 2.1 that the characteristic polynomials for two similar matrices \underline{A} & \underline{B} are the same, hence \underline{A} and \underline{B} have the same eigenvalues, and determinant.

Theorem 2.1:

Necessity: Suppose that there is a set of linearly independent ~~vectors~~ eigenvectors $\{\underline{v}_1, \dots, \underline{v}_n\}$; i.e.,

$$\underline{A}\underline{v}_i = \lambda_i \underline{v}_i, \quad i=1, \dots, n \tag{1}$$

Let $\underline{V} = [\underline{v}_1, \dots, \underline{v}_n]$, and $\underline{D} = \text{Diag}(\lambda_1, \dots, \lambda_n)$. Eq. (1) is rewritten as

$$\underline{A}\underline{V} = \underline{V}\underline{D}$$

Since \underline{V} is nonsingular,

$$\underline{V}^{-1}\underline{A}\underline{V} = \underline{D}$$

* Sufficiency: Suppose that \underline{A} is diagonalizable so that

$$\underline{S}^{-1}\underline{A}\underline{S} = \underline{D} \tag{2}$$

for some nonsingular \underline{S} and diagonal \underline{D} . ^{From} Eq. (2),

$$\underline{A}\underline{S} = \underline{S}\underline{D}$$

or

$$\underline{A}\underline{s}_i = d_i \underline{s}_i$$

where $\underline{S} = [s_1, \dots, s_n]$, $\underline{D} = \text{Diag}(d_1, \dots, d_n)$. By the definition of eigenvalues and eigenvectors, λ_i and s_i are an eigenvalue and s_i eigenvector.

Property 2.3

This property is proved by contradiction. Suppose that $\lambda_i \neq \lambda_k \forall i \neq k$, and that $\{v_1, \dots, v_k\}$ is linearly dependent. Then there exists $\{c_i\}$ such that

$$\sum_{i=1}^k c_i v_i = \underline{0} \tag{1}$$

w.l.o.g. assume that $c_1 \neq 0$ and $c_k \neq 0$. Eq. (1) can be rewritten as

$$\sum_{\substack{i=1 \\ c_i \neq 0}}^k c_i v_i = \underline{0} \tag{2}$$

Pre-multiplying (2) by A , we obtain

$$\sum_{\substack{i=1 \\ c_i \neq 0}}^k c_i \lambda_i v_i = \underline{0} \tag{3}$$

From (2),

$$\begin{aligned} \lambda_k \sum_{\substack{i=1 \\ c_i \neq 0}}^k c_i v_i &= \underline{0} = \sum_{\substack{i=1 \\ c_i \neq 0}}^k c_i \lambda_i v_i \\ \Leftrightarrow \sum_{\substack{i=1 \\ c_i \neq 0}}^{k-1} c_i (\lambda_k - \lambda_i) v_i &= \underline{0} \end{aligned} \tag{4}$$

Repeating the steps in (3) and (4), we end up with

$$c_1 \prod_{\substack{i=1 \\ c_i \neq 0}}^k (\lambda_i - \lambda_1) v_1 = \underline{0} \tag{5}$$

Eq. (5) holds only when $v_1 = \underline{0}$. This contradicts the definition of eigenvector.

Repeated eigenvalues:

When $\text{rank}(A - \lambda I) = n - r$, $\dim \mathcal{N}(A - \lambda I) = r$. Thus, we can find a set of r linearly independent \underline{v} such that

$$(A - \lambda I) \underline{v} = \underline{0}.$$

Property 2.4:

$$\begin{aligned} A \underline{v} &= \lambda \underline{v} \\ \underline{v}^H A \underline{v} &= \lambda \underline{v}^H \underline{v} \end{aligned} \quad (1)$$

LHS of (1):

$$\begin{aligned} \underline{v}^H A \underline{v} &= \sum_i \sum_k v_i^* v_k a_{ik} \\ &= \sum_i |v_i|^2 a_{ii} + \sum_{i>k} v_i^* v_k a_{ik} + \sum_{i<k} v_i^* v_k a_{ik} \\ &= \sum_i |v_i|^2 a_{ii} + \sum_{i>k} v_i^* v_k a_{ik} + \sum_{i>k} (v_i^* v_k a_{ik})^* \\ &= \sum_i |v_i|^2 a_{ii} + 2 \operatorname{Re} \left\{ \sum_{i>k} v_i^* v_k a_{ik} \right\} \end{aligned} \quad (2)$$

Note that (2) is due to $a_{ik} = a_{ki}^*$. Since a_{ii} is real for any i , (2) is real. Since the RHS of (1) is $\lambda \|\underline{v}\|_2^2$, λ cannot be complex.

Property 2.5:

$$A x_i = \lambda_i x_i$$

$$x_k^H A x_i = \lambda_i x_k^H x_i$$

$$(A^H x_k)^H x_i = \lambda_i x_k^H x_i$$

$$(A x_k)^H x_i = \lambda_i x_k^H x_i$$

$$\lambda_k^* x_k^H x_i = \lambda_i x_k^H x_i$$

Since λ_k is real for Hermitian A (cf., Property 2.4),

$$(\lambda_k - \lambda_i) x_k^H x_i = 0.$$

Since $\lambda_k \neq \lambda_i$ for $k \neq i$, $x_k^H x_i = 0$.

Consider the following 2 lemmas:

Lemma 2.1: Given a vector u , there exists a set of vectors $\{y_2, \dots, y_n\}$ such that $[u, y_2, \dots, y_n]$ is a unitary matrix.

We will see that Lemma 2.1 is true, when we study QR decomposition.

(Matrix deflation)

Lemma 2.2 Consider an $n \times n$ matrix A where its eigenvalues and eigenvectors are denoted by λ_i and v_i , respectively. Let S be a nonsingular matrix where its first column is v_1 . Then,

$$S^{-1}AS = \left[\begin{array}{c|c} \lambda_1 & * \\ \hline 0 & A_2 \end{array} \right]. \quad (1)$$

Moreover, the eigenvalues of A_2 are $\lambda_2, \dots, \lambda_n$.

Proof of Lemma 2.2:

$$Av_1 = \lambda_1 v_1$$

$$S^{-1}Av_1 = \lambda_1 S^{-1}v_1$$

$$S^{-1}AS S^{-1}v_1 = \lambda_1 S^{-1}v_1$$

From the fact $S^{-1}S = I$, we have that $S^{-1}v_1 = e_1$. Hence,

$$S^{-1}AS e_1 = \lambda_1 e_1$$

and the matrix structure in (1) follows. Moreover, due to (1),

$$\det(S^{-1}AS) =$$

$$\det(S^{-1}AS - \lambda_1 I) = (\lambda_1 - \lambda_1) \det(A_2 - \lambda_1 I)$$

Hence, $\det(A_2 - \lambda I) = \prod_{i=2}^n (\lambda_i - \lambda)$ where $\det(S^{-1}AS - \lambda I) = \prod_{i=1}^n (\lambda_i - \lambda)$.

Proof of Theorem 2.2:

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Let $\underline{u}_1 = [\underline{u}_1, \underline{y}_2, \dots, \underline{y}_n]$ where \underline{u}_1 is ~~the~~ an eigenvector of A , and $\{\underline{u}_1, \underline{y}_2, \dots, \underline{y}_n\}$ is orthonormal (Lemma 2.1). By Lemma 2,

$$\underline{u}_1^H A \underline{u}_1 = \left[\begin{array}{c|c} \lambda_1 & * \\ \hline 0 & A_2 \end{array} \right] \quad (3)$$

where the eigenvalues of A_2 are $\lambda_2, \dots, \lambda_n$. Apply the procedure in

(3) on A_2 ,

$$\underline{u}_2^H A_2 \underline{u}_2 = \left[\begin{array}{c|c} \lambda_2 & * \\ \hline 0 & A_3 \end{array} \right]$$

By letting

$$\underline{u} = \underline{u}_1 \begin{bmatrix} 1 & 0 \\ 0 & \underline{u}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \underline{u}_3 \end{bmatrix} \dots \begin{bmatrix} \underline{I} & \\ & \underline{u}_{n-1} \end{bmatrix}$$

we obtain

$$\underline{u}^H A \underline{u} = \underline{I}$$

Property 2.6: ~~is~~ Obtain by using the standard result $(\underline{A}\underline{B})^{-1} = \underline{B}^{-1}\underline{A}^{-1}$.

Property 2.7:

Assume that $\lambda_i \neq 0$ for $i=1, 2, \dots, r$, and $\lambda_i = 0$ for $i=r+1, \dots, n$.

The property is proved by considering the dimension of $\mathcal{R}(A)$:

$$\mathcal{R}(A) = \{ y \in \mathbb{C}^n \mid y = Ax, x \in \mathbb{C}^n \}$$

By eigendecomposition we have

$$y = V\Lambda V^H x$$

Since V^H is a basis for \mathbb{C}^n , it is equivalent that

$$\mathcal{R}(A) = \{ y \in \mathbb{C}^n \mid y = V\Lambda \alpha, \alpha \in \mathbb{C}^n \}$$

Now,

$$\begin{aligned} y &= V\Lambda \alpha \\ &= \sum_{i=1}^r v_i \lambda_i \alpha_i \end{aligned}$$

Thus, $\mathcal{R}(A) = \text{span}[v_1, \dots, v_r]$ and it follows that

$$\text{rank}(A) = \dim \mathcal{R}(A) = r.$$

Matrix 2-norm

$$\begin{aligned}\|A\|_2^2 &\equiv \max_{\|x\|_2=1} \|Ax\|_2^2 \\ &= \max_{\|x\|_2=1} x^H A^H A x\end{aligned}$$

$A^H A$ is a Hermitian, and hence it has an eigendecomposition

$$A^H A = U \Lambda U^H$$

Then,

$$\begin{aligned}x^H A^H A x &= x^H U \Lambda U^H x \\ &= \sum_{i=1}^n \lambda_i |u_i^H x|^2 \\ &\leq \sum_{i=1}^n \lambda_{\max} |u_i^H x|^2\end{aligned}$$

$$= \lambda_{\max} \|U^H x\|_2^2$$

$$= \lambda_{\max} \|x\|_2^2$$

$$= \lambda_{\max}$$

It can be verified that, for $\|x\|_2=1$

Equality holds when $x = u_{\max}$. Hence, $\|A\|_2^2 = \lambda_{\max}$.