COM521500
Math. Methods for SP I
Lecture 11: Matrix Equations and the Kronecker Product

Motivation

In this lecture we study several linear operators, namely the **Kronecker product**, the **vectorization**, and the **Kronecker sum**.

They are very useful in solving seemingly hard matrix eqns., such as solving

$$\mathbf{X}\mathbf{A} + \mathbf{A}^H\mathbf{X} = \mathbf{H}$$

for $\mathbf{X}$ given $\mathbf{A}$ and $\mathbf{H}$.

Kronecker Product

The Kronecker product of two matrices $\mathbf{A}$ and $\mathbf{B}$ are defined to be

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & & a_{2n}\mathbf{B} \\ \vdots & & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{bmatrix}$$

Some elementary properties for the Kronecker product:

1. $\mathbf{A} \otimes (\alpha \mathbf{B}) = \alpha \mathbf{A} \otimes \mathbf{B}$.
2. (distributive)

$$(\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}$$
$$\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}$$

3. (associative) $\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}$.
4. $\mathbf{0}_{mn} = \mathbf{0}_m \otimes \mathbf{0}_n$, $\mathbf{I}_{mn} = \mathbf{I}_m \otimes \mathbf{I}_n$.
5. $(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T$, $(\mathbf{A} \otimes \mathbf{B})^H = \mathbf{A}^H \otimes \mathbf{B}^H$.

6. (mixed product rule)

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD})$$

for $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ of appropriate matrix dimensions.

7. Suppose that $\mathbf{A} \in \mathbb{C}^{m \times m}$ and $\mathbf{B} \in \mathbb{C}^{n \times n}$ are nonsingular.

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$$

Property 7 can be shown using Property 6:

$$\begin{aligned}(\mathbf{A}^{-1} \otimes \mathbf{B}^{-1})(\mathbf{A} \otimes \mathbf{B}) &= (\mathbf{A}^{-1}\mathbf{A}) \otimes (\mathbf{B}^{-1}\mathbf{B}) \\ &= \mathbf{I}_m \otimes \mathbf{I}_n = \mathbf{I}_{mn}\end{aligned}$$

The Kronecker product is not commutative in general; i.e., $\mathbf{A} \otimes \mathbf{B} = \mathbf{B} \otimes \mathbf{A}$ is not true except for special cases such as $\mathbf{A} = a$ & $\mathbf{B} = b$. However,

8. There exist permutation matrices $\mathbf{U}_1$ and $\mathbf{U}_2$ such that

$$\mathbf{U}_1(\mathbf{A} \otimes \mathbf{B})\mathbf{U}_2 = \mathbf{B} \otimes \mathbf{A}$$

There is a straightforward correspondence between the eigen-eqns. of $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{A}$, $\mathbf{B}$.

Theorem 11.1 Let $\mathbf{A} \in \mathbb{C}^{m \times m}$, & $\mathbf{B} \in \mathbb{C}^{n \times n}$. Let $\{\lambda_i, \mathbf{x}_i\}_{i=1}^m$ be the set of m eigen-pairs of $\mathbf{A}$, and $\{\mu_i, \mathbf{y}_i\}_{i=1}^n$ be the set of n eigen-pairs of $\mathbf{B}$. The set of mn eigen-pairs of $\mathbf{A} \otimes \mathbf{B}$ is given by

$$\{\lambda_i \mu_j, \mathbf{x}_i \otimes \mathbf{y}_j\}_{i=1,\ldots,m,\ j=1,\ldots,n}$$

From Theorem 11.1 it follows that

9. $\det(\mathbf{A} \otimes \mathbf{B}) = [\det(\mathbf{A})]^n [\det(\mathbf{B})]^m$.
10. $\mathrm{tr}\{\mathbf{A} \otimes \mathbf{B}\} = \mathrm{tr}\{\mathbf{A}\}\mathrm{tr}\{\mathbf{B}\}$.
11. If $\mathbf{A}$ & $\mathbf{B}$ are (Hermitian) PSD, then $\mathbf{A} \otimes \mathbf{B}$ is PSD.

Example: Hadamard Matrix

Let

$$\mathbf{H}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

This matrix is orthogonal.

We can construct a 4×4 matrix by

$$\begin{aligned} \mathbf{H}_4 &= \mathbf{H}_2 \otimes \mathbf{H}_2 \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{aligned}$$

Is $\mathbf{H}_4$ orthogonal? Yes, because
$\mathbf{H}_4\mathbf{H}_4^T = (\mathbf{H}_2 \otimes \mathbf{H}_2)(\mathbf{H}_2^T \otimes \mathbf{H}_2^T) = (\mathbf{H}_2\mathbf{H}_2^T \otimes \mathbf{H}_2\mathbf{H}_2^T) = \mathbf{I}$.

We can obtain $\mathbf{H}_n$ for any even n in a similar way.

Vectorization

Let $\mathbf{A} = [\ \mathbf{a}_1, \ldots, \mathbf{a}_n\ ]$.

$$\text{vec}(\mathbf{A}) = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix}$$

The vectorization operation stacks the columns of a matrix to form a column vector.

An important property:

$$\text{vec}(\mathbf{AXB}) = (\mathbf{B}^T \otimes \mathbf{A})\text{vec}(\mathbf{X})$$

Special cases of this property are

$$\text{vec}(\mathbf{AX}) = (\mathbf{I} \otimes \mathbf{A})\text{vec}(\mathbf{X})$$
$$\text{vec}(\mathbf{XA}) = (\mathbf{A}^T \otimes \mathbf{I})\text{vec}(\mathbf{X})$$

Example: Space-Time Block Coding

Let M & N be no. of tx and rx antennas. Let T be the code length.

Signal model:

$$\mathbf{Y} = \mathbf{H}\mathbf{C}(\mathbf{s}) + \mathbf{V}$$

where

$\mathbf{Y} \in \mathbb{C}^{M \times T}$	received code matrix
$\mathbf{H} \in \mathbb{C}^{M \times N}$	channel matrix
$\mathbf{C}(\mathbf{s}) \in \mathbb{C}^{N \times T}$	transmitted space-time block code (STBC)

The tx STBC has a linear dispersion structure

$$\mathbf{C}(\mathbf{s}) = \sum_{k=1}^{K} \mathbf{X}_k s_k$$

where $\mathbf{X}_k \in \mathbb{C}^{N \times T}$ are its basis matrices.

Our aim is to estimate $\mathbf{s}$ from $\mathbf{Y}$.

Vectorizing the signal model yields

$$\text{vec}(\mathbf{Y}) = (\mathbf{I}_T \otimes \mathbf{H})\text{vec}(\mathbf{C}(\mathbf{s})) + \text{vec}(\mathbf{V})$$

Moreover,

$$\begin{aligned}\text{vec}(\mathbf{C}(\mathbf{s})) &= \sum_{k=1}^{K} \text{vec}(\mathbf{X}_k) s_k \\ &= \underbrace{\left[\ \text{vec}(\mathbf{X}_1), \ldots, \text{vec}(\mathbf{X}_K)\ \right]}_{\boldsymbol{\mathcal{X}}} \mathbf{s}\end{aligned}$$

Hence, we obtain a familiar linear LS model:

$$\text{vec}(\mathbf{Y}) = (\mathbf{I}_T \otimes \mathbf{H})\boldsymbol{\mathcal{X}}\mathbf{s} + \text{vec}(\mathbf{V})$$

which allows us to use LS to estimate s.

Kronecker Sum

The Kronecker sum is motivated by the necessity of solving this problem

$$\mathbf{AX} + \mathbf{XB} = \mathbf{C}$$

where $\mathbf{A} \in \mathbb{C}^{n\times n}$, $\mathbf{B} \in \mathbb{C}^{m\times m}$, & $\mathbf{C}, \mathbf{X} \in \mathbb{C}^{n\times m}$.

By applying vectorization,

$$(\mathbf{I}_m \otimes \mathbf{A})\text{vec}(\mathbf{X}) + (\mathbf{B}^T \otimes \mathbf{I}_n)\text{vec}(\mathbf{X}) = \text{vec}(\mathbf{C})$$

The Kronecker sum for two square matrices $\mathbf{A} \in \mathbb{C}^{n \times n}$, $\mathbf{B} \in \mathbb{C}^{m \times m}$ are defined to be

$$\mathbf{A} \oplus \mathbf{B} = (\mathbf{I}_m \otimes \mathbf{A}) + (\mathbf{B} \otimes \mathbf{I}_n)$$

Theorem 11.2 Let $\{\lambda_i, \mathbf{x}_i\}_{i=1}^{n}$ be the set of m eigen-pairs of $\mathbf{A}$, and $\{\mu_i, \mathbf{y}_i\}_{i=1}^{m}$ be the set of n eigen-pairs of $\mathbf{B}$. The set of mn eigen-pairs of $\mathbf{A} \otimes \mathbf{B}$ is given by

$$\{\lambda_i + \mu_j, \mathbf{y}_j \otimes \mathbf{x}_i\}_{i=1,\ldots,n,\ j=1,\ldots,m}$$

Theorem 11.3 The matrix equations

$$\mathbf{AX} + \mathbf{XB} = \mathbf{C}$$

has a unique solution for every given $\mathbf{C}$ if and only if

$$\lambda_i \neq -\mu_j \qquad (*)$$

for all i, j.

The idea of this theorem is as follows: If $(*)$ can be satisfied, then from Theorem 11.2 there exist a zero eigenvalue implying $\mathbf{A} \oplus \mathbf{B}^T$ is singular.

Consider the special case

$$\mathbf{A}^H\mathbf{X} + \mathbf{X}\mathbf{A} = \mathbf{C}$$

which are known as the Lyapunov equations. From Theorem 11.3, it has a unique solution if

$$\lambda_i \neq -\lambda_j^*$$

for all i, j.

If $\mathbf{A}$ is PD such that λ_i are real and $+$ve, then the Lyapunov equations always have a unique solution.